

2013

Generalizing Holographic Superconductors

Patrick King
College of William and Mary

Follow this and additional works at: <http://publish.wm.edu/honorsthesis>

Recommended Citation

King, Patrick, "Generalizing Holographic Superconductors" (2013). *Undergraduate Honors Theses*. Paper 626.
<http://publish.wm.edu/honorsthesis/626>

This Honors Thesis is brought to you for free and open access by the Theses, Dissertations, & Master Projects at W&M Publish. It has been accepted for inclusion in Undergraduate Honors Theses by an authorized administrator of W&M Publish. For more information, please contact wmpublish@wm.edu.

Generalizing Holographic Superconductors

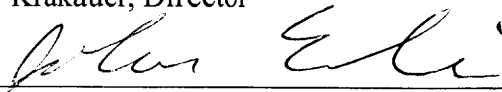
A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science in Physics from
The College of William and Mary

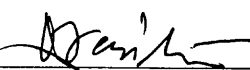
by

Patrick King

Accepted for Honors
(Honors)


Henry Krakauer, Director


Joshua Erlich, Advisor


Daniel Vasiliu

Williamsburg, VA
April 25, 2013

Generalizing Holographic Superconductors

Patrick King

Department of Physics,
College of William and Mary,
Williamsburg, VA 23187-8795, USA

Email: pkking@email.wm.edu

April 12, 2013

Abstract

We consider a family of holographically dual models of superconductivity to test the robustness of holographic superconductor models in general. Following the treatment of Hartnoll et al. and Albrecht et al., we introduce the basic holographic model and then develop a family of similar models by varying a parameter in the spacetime metric of the holographic theory. We then calculate several observables and compare with actual superconductor observables. This helps determine whether these holographic models actually explain superconductivity or if the agreement between the theories and experiment is a coincidence.

Contents

1	Introduction	1
1.1	Motivation	1
1.2	Superconductivity	2
1.3	Holography	3
1.4	Holographic Superconductors	6
2	Generalized Holographic Superconductor Models	9
2.1	The Basic Model	9
2.2	Generalization	11
3	Calculation of Observables	13
3.1	The Cooper Pair Condensate	13
3.2	The Conductivity	15
4	Conclusion	19
4.1	Future Work	19
4.2	Acknowledgements	20
A	Determination of the Hawking Temperature	21

List of Figures

1.1	Simplified view of an event horizon. The arrows represent the possible trajectories of the particle through space and time.	5
1.2	Superconducting geometry of cuprate superconductors.	7
3.1	Numerical results for calculating the cooper pair condensate.	14
3.2	Numerical results for calculating the conductivity. Note the delta-function at the origin for all three solutions.	16

Chapter 1

Introduction

1.1 Motivation

Superconductivity is a phenomenon which has been subject to intensive study by the physics community since its discovery. The development of a full and complete theory of superconductivity would provide deep insights into the physics of condensed matter at the fundamental level and also might provide a roadmap to develop novel superconducting materials. We are fortunate in having a very successful theory of superconductivity, known as BCS theory after its inventors (Bardeen, Cooper, and Schrieffer) [1]. BCS theory has successfully predicted the superconducting behavior of so called low-temperature superconductors. However, BCS theory alone cannot describe materials who superconduct at higher temperatures; this has led to widespread effort to develop theories which accurately describe these materials.

Indeed, this effort has not escaped the notice of particle physicists, who have developed theories of superconductivity utilizing the principle of holography [2] [3] [4], which is the subject of our work. These theories develop a model of superconductivity which describes the phenomenon as a higher dimensional theory, which is “dual” to the ordinary dimensional theory. These efforts are inspired by other instances of holography found in

particle physics, such as the study of black hole thermodynamics and string theory. The models developed by Hartnoll et al. [2] have yielded some interesting results which seem to match properties of high-temperature superconducting behavior, naturally motivating further work with these models.

Essentially our research goals were to investigate possible generalizations of the work of Hartnoll et al. [2] For simplicity, a specific spacetime metric was used to create the holographic dual theory. We decided to consider the same model making procedure with several other valid spacetime metrics. If these models of superconductivity offer a compelling theoretical description of superconductivity (in particular high-temperature superconducting behavior) then we would expect such models to robustly predict high-temperature superconductor observables under several different equivalent regimes.

1.2 Superconductivity

A brief review of superconductivity is useful to frame our discussion. Superconductivity is a phenomenon in which a conductor displays two unusual properties: perfect conductivity and the Meissner effect [1]. Conductivity is a property of a material which describes how easily currents flow in that material; a material with perfect conductivity allows direct currents to flow unimpeded. (Note: this does not mean that alternating currents flow without impedance; there is a nonvanishing impedance associated with these currents even in perfect conductors, especially at very high frequencies.) However, a material possessing only perfect conductivity is simply known as a perfect conductor; a superconductor must have this as well as perfect diamagnetism as mentioned before. The Meissner effect is the exclusion or expulsion of all magnetic fields within the conductor. This is not a consequence of perfect conductivity but is a distinct phenomenon. To date, the only perfect conductors discovered have been superconductors [1].

Superconducting materials undergo a phase transition with a characteristic *critical temperature* under which the material enters a state with these two fundamental proper-

ties. Superconductors are grouped into two types: Type-I and Type-II. (The distinction between these two types is related to the ratio between the London penetration depth and the superconducting coherence length; this is discussed in more depth in Tinkham [1].) Type-I superconductors all have critical temperatures at or below 30 K; their behavior is well described by the BCS theory. In fact, the 30 K limit comes from BCS theory, and before the discovery of Type-II superconductivity, 30 K was thought to be the absolute upper limit for superconducting states. However, some, but not all, Type-II superconductors operate above the 30 K limit, which naturally indicates a limitation of the BCS theory. Alone, it cannot accurately describe such materials [1]. (Type-II superconductors also display unusual behavior not seen in Type-I superconductors, such as the presence of magnetic vortices known as Abrikosov vortices into the interior of the superconductor.)

We understand the conduction of electricity in materials as essentially an electron gas moving in a fixed lattice of atoms. When the gas flows in the lattice, some of the electrons collide with individual atoms and this transfers heat to the conductor. The electrons are scattered by the atoms, and so the lattice resists the flow of the electron gas. This is the origin of electrical resistance. According to BCS theory, however, materials in the superconducting state behave differently. Pairs of electrons condense into a quasiparticle known as a Cooper pair; this occurs due to the electrons interacting with phonons in the lattice [1]. These pairs are able to flow as a superfluid in the lattice of the superconductor, meaning the electrons will flow with no resistance. The superconducting behavior is thus directly related to interactions with the lattice itself.

1.3 Holography

Holography is a broad term encompassing a variety of phenomena in which the physics of a system is described equivalently in two different theories, each with differing dimensionality. For example, the physics of a black hole, an object in 3+1 dimensions, can be described entirely with the information on the horizon of the black hole, the boundary of

the object which is in 2+1 dimensions [6]. (The convention “3+1” refers to the number of spatial dimensions plus the number of time dimensions; this reflects the usage found in the literature.) In a slightly more general case, the Maldacena conjecture [5], alternatively known as the AdS/CFT correspondence, relates a theory in higher-dimensional Anti-de Sitter space with a field theory on the conformal boundary of this space. (This is a conjectured correspondence, strongly motivated by work in string theory, but is not proven [6].)

The case of black hole thermodynamics merits some more in depth discussion. Work done by both Hawking and Bekenstein has demonstrated that black holes have thermodynamic properties, such as temperature and entropy. This is at first glance quite striking, because black holes are, naively, purely gravitational constructs; black hole solutions are determined by purely geometric equations found in general relativity. However, combining the physics of the event horizon with quantum effects of the vacuum leads to the interesting phenomenon of Hawking radiation [6]. Consider an event horizon in the vacuum, as in figure 1.1.

Any particle found on side one must fall into the black hole, and a particle found on side two may escape. Ordinarily, in a perfect classical (relativistic) universe, absolutely nothing can escape from the black hole; anything that crosses the event horizon is forever trapped. However, we live in a quantum universe. Not only do we assign a nonzero probability of a particle tunneling through this barrier, but we also have observed the phenomenon of pair production. Sometimes, energy is “borrowed” from the vacuum to allow a particle-antiparticle pair to be produced briefly; this is allowed due to the energy-time uncertainty relation, given by [7]

$$\Delta E \Delta t \geq \frac{\hbar}{2}. \tag{1.1}$$

These virtual particles wink in and out of existence throughout the vacuum. (Indeed, this phenomenon is better understood using creation and annihilation operators found in

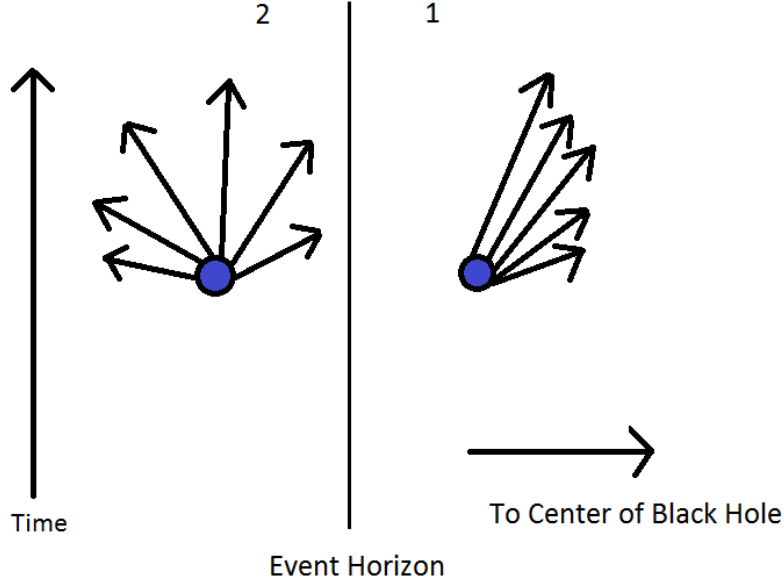


Figure 1.1: Simplified view of an event horizon. The arrows represent the possible trajectories of the particle through space and time.

second quantization.) Of course, combining this with the physics of the event horizon, we have a potential issue. What happens when a pair is produced across the event horizon, with one particle found on one side and its antiparticle found on the other side? Naturally, one particle must be lost to the black hole, but the other particle might not be. But now, the lost particle cannot annihilate its antipartner, and so the other particle is no longer virtual, and the energy is no longer borrowed. When we recall that energy has to be conserved, then we realize that the energy has to come from somewhere. The answer is that the energy comes from the black hole itself, and that the black hole has “radiated” this particle. This black hole must then have a temperature, and an entropy. In fact, there are several laws of black hole thermodynamics [6], much like the laws of ordinary thermodynamics, which relate statistical and gravitational quantities. We find that the entropy of a black hole, which we would naively expect to be proportional to the volume of microstates available to the system, is instead proportional to the area of the

black hole horizon. We thus have a correspondence between two physical regimes, which is the essence of the holographic principle.

Another example of holography is the hypothesized AdS/CFT correspondence as mentioned above, found in string theory. AdS/CFT refers to a correspondence between a string theory in anti-de Sitter space and a quantum field theory on its conformal boundary. This was first conjectured by Maldacena [5]; in fact, it is known that a theory of quantum gravity is holographic, and so if string theory indeed provides a quantum theory of gravity we should expect holography [6]. AdS/CFT provides a “dictionary” that relates specific quantities in the two theories.

1.4 Holographic Superconductors

Thus the program we are pursuing is to use this AdS/CFT dictionary to develop a theory of superconductivity using holographic techniques. This is precisely what is done in Hartnoll et al. [2], and there is more elaboration in their review paper [4]. We will be following their model, in which we introduce a charged complex scalar field (representing cooper pairing) and the Maxwell field on an anti-de Sitter space black hole metric. We will be working in the continuum, rather than a deconstructed approach, which is pursued in Albrecht et al. [3]

The particular model we are studying will produce a theory in 3+1 spatial dimensions, which is itself what we experience in everyday life. The holographic dual model will then be in 2+1 dimensions; that is, we will be describing superconductivity in the plane in time. As it happens, this is the type of behavior exhibited by cuprate superconductors, which gives us a direct example of a superconductor to check our model [2]. The superconducting geometry is shown in figure 1.2. However, rather than just considering the AdS-Schwarzschild solution, we are instead interested in applying this technique to other spacetimes that have similar, but not identical, behavior. The family of solutions, described below, all have horizon behavior. If this model of high temperature supercon-

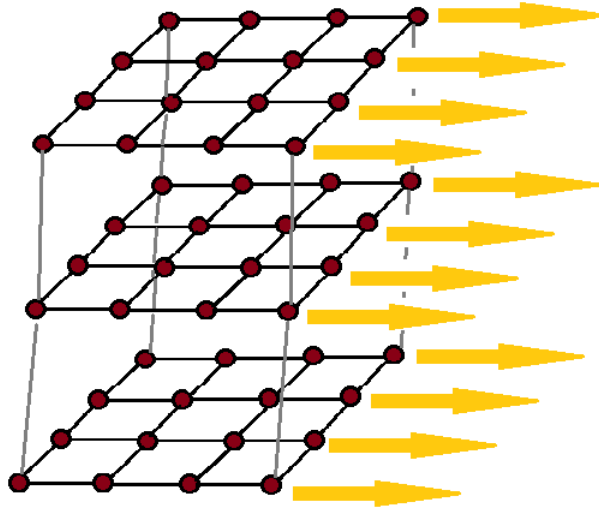


Figure 1.2: Superconducting geometry of cuprate superconductors.

ductivity is robust - that is, if it is not just coincidental - we should see behavior in these models similar to the original model. In particular, if these holographic models offer a compelling description of high-temperature superconductivity - that is, if high-temperature superconducting behavior is a result of holographic effects - then these models should predict the same value for the superconducting gap, a well known observable whose value is wrongly predicted by conventional BCS theory. If the model is purely coincidental, and the high-temperature description is not due to holographic effects but peculiar aspects of the spacetime metrics, then we would not expect our models to provide similar results to the original model.

Chapter 2

Generalized Holographic Superconductor Models

2.1 The Basic Model

We will first give an overview of the holographic superconductor model first given by Hartnoll et al. [2] The first step is specifying the spacetime metric, which is given by the planar anti-de Sitter black hole solution:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_2^2. \quad (2.1)$$

Here, $d\Omega_2^2$ refers to the polar angular dependence; we are working in 3+1 dimensions, so the angular dependence is two-dimensional. As in [2] and [3] we work in the limit where this spacetime is appropriate, neglecting the backreaction of the charge density of the geometry. (If we were to include the backreaction, the Reissner-Nordstrom spacetime geometry is more appropriate.) The function $f(r)$ is given by [3]

$$f(r) = \frac{r^2}{L^2} \left(1 - \frac{\mu}{r^3}\right). \quad (2.2)$$

The parameters μ and L are the mass and length parameters of the black hole. This 3+1

theory will be the dual to a 2+1 theory of superconductivity, consistent with the program outlined previously.

As described in Appendix A, we know that the horizon of the black hole is given by $r_H = \sqrt[3]{\mu}$, and the temperature of the black hole is given by

$$T = \frac{3r_H}{4\pi L^2}. \quad (2.3)$$

We note that the r coordinate runs from r_H to infinity, since the spacetime metric does not describe space inside the horizon. The position $r = \infty$ is referred to as the UV boundary, while $r = r_H$ is referred to as the horizon or IR boundary.

To introduce superconductivity, we must include a charged complex scalar field through a lagrangian density function [3]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |(\partial_\mu - iA_\mu)\psi|^2 - m^2|\psi|^2. \quad (2.4)$$

Here A_μ is the electromagnetic four-vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic energy tensor, ψ is the charged complex scalar field, and the m^2 corresponds to a cooper pair operator of dimension 1 or 2, which is explained below [3]. (The indices μ and ν run over the 3+1 dimensions of the theory.)

$$m^2 = \frac{2}{L^2}. \quad (2.5)$$

This lagrangian density is integrated under the action

$$S = \int \mathcal{L}\sqrt{g} d^4x. \quad (2.6)$$

Extremizing this, and identifying the quantity $\phi = A_0$ leads to the coupled system of differential equations [3]

$$\psi'' + \left(\frac{f'(r)}{f(r)} + \frac{2}{r}\right)\psi' + \frac{\phi^2}{f(r)^2}\psi - \frac{m^2}{f(r)}\psi = 0, \quad (2.7)$$

$$\phi'' + \frac{2}{r}\phi' - \frac{2|\psi|^2}{f(r)} = 0. \quad (2.8)$$

These equations can be solved with appropriate boundary conditions (discussed in the next section) to determine the fields ψ and ϕ .

At the UV boundary (which corresponds to $r \rightarrow \infty$), we know that ψ and ϕ behave as

$$\psi = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + \dots \quad (2.9)$$

$$\phi = \mu - \frac{\rho}{r} + \dots \quad (2.10)$$

We identify μ as a chemical potential and ρ as a charge density, in accord with the AdS/CFT dictionary [2]. The quantities $\psi^{(1)}$ and $\psi^{(2)}$ are normalizable, and so we can assert a boundary condition that one of them vanishes. This is described below.

2.2 Generalization

Our task now is to generalize the model described above. Rather than considering the AdS-Schwarzschild solution, which was chosen for simplicity, we want to now complicate the model slightly by changing the form of $f(r)$ to represent a slightly different spacetime metric. Our choice of new $f(r)$ should preserve black hole behavior and a horizon, so as to still observe holographic effects. We decided to choose

$$f_p(r) = \frac{r^2}{L^2} \left(1 - \frac{\mu}{r^p}\right). \quad (2.11)$$

In essence, we have changed the form of $f(r)$ quite simply to one dependent on parameter p . In order to preserve holographic effects we should choose $p \geq 3$; thus we recover the previous model as a specific case of this model. The parameter μ still yields the position of the horizon, but with the form $r_H = \sqrt[p]{\mu}$. As in Appendix A, we find the temperature

$$T = \frac{pr_H}{4\pi L^2}. \quad (2.12)$$

We still retain the same Lagrangian density, and since the differential equations 2.6 and 2.7 keep the form of $f(r)$ unspecified, we simply change our $f(r)$ in our numerics to calculate results for different models.

Chapter 3

Calculation of Observables

3.1 The Cooper Pair Condensate

The cooper pair condensate can be identified directly with the field ψ which is specified by the differential equations 2.6 and 2.7. This relationship is dictated by the AdS/CFT correspondence [2]:

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\psi^{(i)}. \tag{3.1}$$

As mentioned above, both $\psi^{(1)}$ and $\psi^{(2)}$ are normalizable and so we are free to choose one of them to vanish; we choose $\psi^{(1)}$ to do so. The reason why this is true is that AdS/CFT normally finds two independent solutions, only one is typically normalizable, which is taken as the condensate. However, due to the r -dependence both terms have finite action, and the interpretation is that AdS/CFT allows us the freedom of choosing which of the two independent solutions is the condensate and which is the source. As we have incorporated no sources in our model, we then set the source term to vanish.

We should now discuss the critical temperature of the superconductor. As with the rules of AdS/CFT, we identify the temperature of the superconductor with the temperature of this black hole geometry. We have also calculated the temperature as a function

of the horizon, as given by equation 2.12. The critical temperature is defined as the temperature at which superconductivity is destroyed; thus we should expect the cooper pair condensate to vanish at the critical temperature. This is where we define the critical temperature - the radius (and thus temperature) at which the cooper pair condensate vanishes [2].

We should now dictate the other boundary conditions. We would like ϕ to have finite norm at the horizon, so we specify that $\phi(r_H) = 0$ [2]. This then specifies boundary behavior at the horizon for ψ to be specified as $\psi(r_H) = -\frac{3r_H}{2}\psi'(r_H)$. [2] Finally, at the UV boundary we expect to recover the chemical potential by the AdS/CFT dictionary, so we specify that $\phi(\infty) = \mu$. [3] Taken together, we have all the boundary conditions we need to solve the equation for $\psi^{(2)}$ and thus the cooper pair condensate.

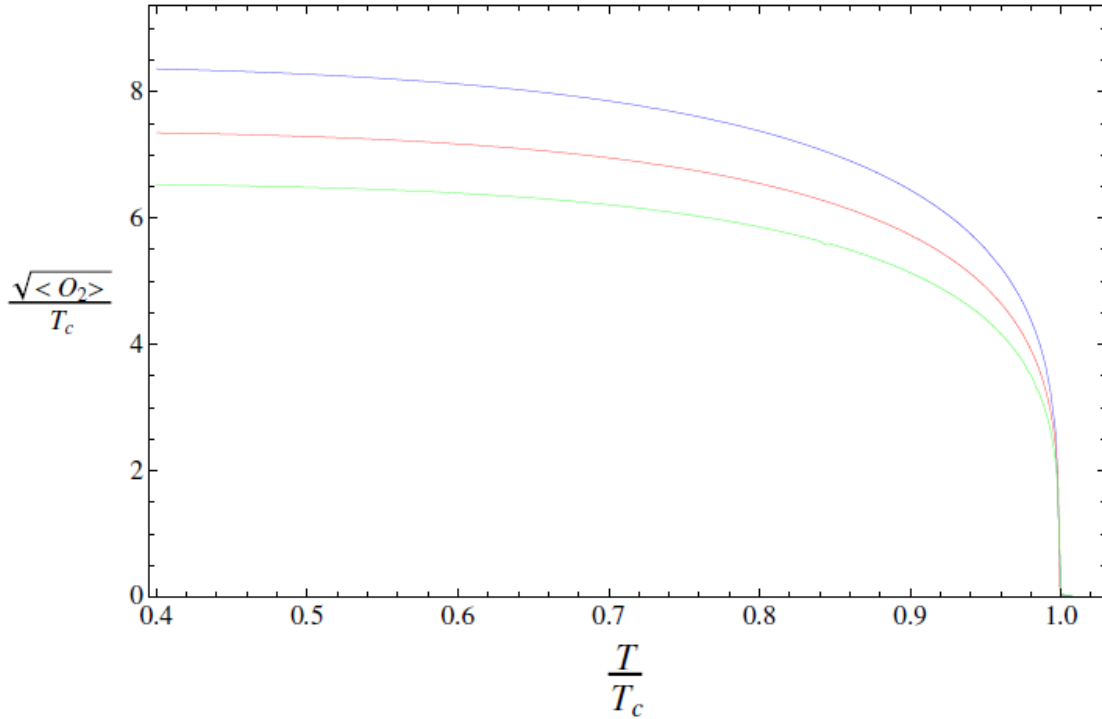


Figure 3.1: Numerical results for calculating the cooper pair condensate.

To compute the fields ψ and ϕ we rely on numerical methods for calculation, as analytic

solutions do not exist. This is not surprising given the complicated nature of these coupled differential equations. The numerical method we used to solve these equations is the shooting method in which this boundary value problem is converted to an initial value problem. (This was implemented in Mathematica.) The results for $p = 3$, $p = 3.5$, and $p = 4$ are shown in figure 3.1, with colors blue, red, and green respectively. The plot is scaled with the critical temperature.

The critical temperature for each of these parameters p is given below:

$$T_c(p = 3) = 0.119\rho^{1/2} \quad (3.2)$$

$$T_c(p = 3.5) = 0.136\rho^{1/2} \quad (3.3)$$

$$T_c(p = 4) = 0.153\rho^{1/2} \quad (3.4)$$

If we compare to Hartnoll et al. [2], we find not only that we recover their result for $p = 3$ but also that the behavior of solutions $p = 3.5$ and $p = 4$ is similar.

3.2 The Conductivity

Calculating the conductivity relies on Ohm's law, given (in one dimension) by

$$J(\omega) = \sigma(\omega)E(\omega). \quad (3.5)$$

That is, the current density is proportional to the applied electric field. As indicated, in general the conductivity has a frequency dependent response, which is a known observable that can be found experimentally. Hence, we should study the response to an oscillating electric field. If we let $A_r = 0$, and choose one direction x in the angular dependence, letting the other direction $A_y = 0$, we can study

$$A_x(t, r) = e^{-i\omega t} A(r). \quad (3.6)$$

The equation of motion for A_x is given by [3]

$$-\frac{\omega^2}{f(r)}A(r) - \frac{d}{dr}(f(r)A'(r)) + 2A(r)|\psi(r)|^2 = 0. \quad (3.7)$$

In the UV limit, A_x has the the form [3]

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r} + \dots \quad (3.8)$$

As in Albrecht et al. [3], we identify E_x and J_x as

$$E_x = -\partial_t A_x|_{r \rightarrow \infty} = i\omega A_x|_{r \rightarrow \infty}, \quad (3.9)$$

$$J_x = A_x^{(1)} = -r^2 \partial_r A_x|_{r \rightarrow \infty}. \quad (3.10)$$

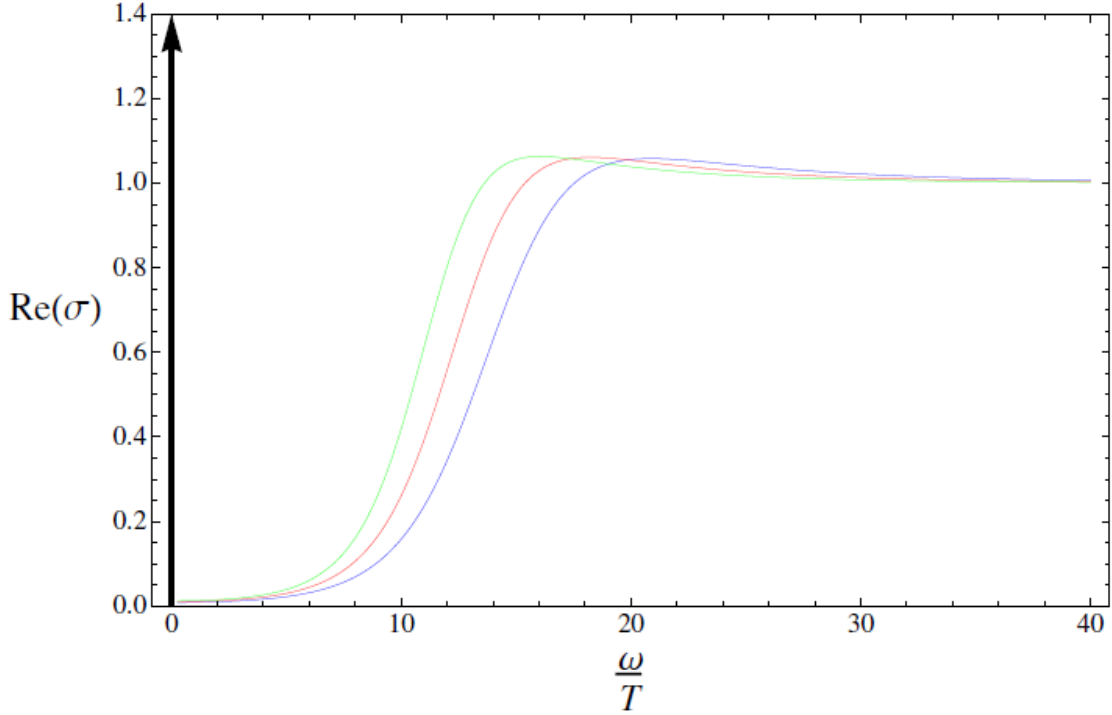


Figure 3.2: Numerical results for calculating the conductivity. Note the delta-function at the origin for all three solutions.

Thus the conductivity is identified as [3]

$$\sigma = \frac{J_x}{E_x} = \frac{-r^2 A'(r)}{i\omega A(r)} \Big|_{r \rightarrow \infty}. \quad (3.11)$$

This can also be numerically calculated, as in Albrecht et al. [3] the delta-function at the origin is a consequence of the Kramer-Kronig relations: there is a pole in the imaginary part of the conductivity which corresponds to a delta-function in the real part. This is fitting given that superconducting behavior is a delta-function for DC (zero frequency) currents. The result for $p = 3$, $p = 3.5$, and $p = 4$ is shown in figure 3.2, again with blue, red, and green plots respectively. (The ratio T/T_c is 0.53.)

We can indeed go further and calculate the superconducting gap from this plot. We know that $Re(\sigma)$ should follow a behavior as

$$Re(\sigma) \propto e^{-\Delta/T}. \quad (3.12)$$

Where Δ is the superconducting gap. For each parameter p we can identify $\Delta = C_p \langle \mathcal{O}_2 \rangle$. Fitting these plots, we obtain:

$$\Delta_3 = 0.50 \sqrt{\langle \mathcal{O}_2 \rangle} \quad (3.13)$$

$$\Delta_{3.5} = 0.55 \sqrt{\langle \mathcal{O}_2 \rangle} \quad (3.14)$$

$$\Delta_4 = 0.60 \sqrt{\langle \mathcal{O}_2 \rangle}. \quad (3.15)$$

Utilizing the data we calculated for the cooper pair condensate (at $T/T_c = 0.53$) and the fact that $\Delta = C_p \langle \mathcal{O}_2 \rangle$ gives us the ratio of the superconducting gap to T_c . We quote 2Δ as in Hartnoll et al. [2]:

$$2\Delta_3 = 8.40T_c \quad (3.16)$$

$$2\Delta_{3.5} = 7.92T_c \quad (3.17)$$

$$2\Delta_4 = 7.68T_c \quad (3.18)$$

While still improvements over the standard BCS prediction of $2\Delta_{BCS} = 3.54T_c$, these results suggest that the holographic effects do not consistently predict a high 2Δ value, and so holographic effects might not be directly responsible for high-temperature superconductivity.

Chapter 4

Conclusion

We have checked the behavior of a class of holographic superconducting models, which appears to reflect similar behavior to the original model as given by Hartnoll et al. [2]. The models appear to have passed our tests of robustness, that is, the behavior of the models is consistent across several parameters of p . The similar qualitative behavior lends support to the idea that holographic models of superconductivity are indeed useful to study. However, we did not find a consistently high value for 2Δ and so holographic effects might not be directly responsible for high-temperature superconductivity.

4.1 Future Work

Our results are not necessarily conclusive evidence that holography is not responsible for high-temperature superconducting behavior. Our assumption that these different spacetime metrics should produce the same superconducting gap behavior might be wrong. After all, spacetime is not only an active participant in our model but also the stage, so to speak, and we should be careful to check our conclusions with further study.

There are several other opportunities for future work starting from this project. In the immediate short term, we can further check the holographic behavior for additional

values of p , potentially determining if there is any bifurcation occurring when p is varied. We can also check different functional forms of $f(r)$ for further robustness testing. In addition, applying this generalization procedure to deconstructed models as in Albrecht et al. [3] is currently being pursued.

We can also start from a different spacetime metric altogether; the Reissner-Nordstrom metric is a natural choice, in which we do not neglect the backreaction of the charge density on spacetime. Additionally the Kerr metric might be interesting to study, since we know how superconductivity as a phenomenon behaves under rotation. Combining these approaches with the Kerr-Newman metric might yield interesting results. Finally, approaching superconductivity from a higher dimensional theory, such as describing 3+1 dimensional superconductivity with a 4+1 dimensional theory is another research avenue.

4.2 Acknowledgements

I would first like to thank my advisor, Professor Joshua Erlich of the William & Mary Department of Physics for his guidance and direction. I would like to thank Zhen Wang, also of the Department of Physics, for his expertise and assistance, particularly with his numerical work without which this work would be quite incomplete. I would like to thank Professor Henry Krakauer for directing the senior research program. I would like to thank the William & Mary Department of Physics in general for providing an excellent research program and resources. I would finally like to thank the Dorothy Pruitt Babcock Memorial Research Scholarship for generously funding my work in the Summer of 2012, which led to the work done this year.

Appendix A

Determination of the Hawking Temperature

For completeness we show the complete derivation of the Hawking temperature from the black hole spacetime metric itself. (This follows the treatment found in [8] [9] [10].) We are given the general anti-de Sitter Schwarzschild black hole solution

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_2^2. \quad (\text{A.1})$$

Where $f(r)$ is a function of r which determines the specific Schwarzschild black hole we are looking at. For some power p , we have a family of AdS/Schwarzschild solutions, with

$$f_p(r) = \frac{r^2}{L^2} \left(1 - \frac{\mu}{r^p}\right). \quad (\text{A.2})$$

In terms of the parameter μ , we find the position of the horizon r_H by demanding that $f_p(r_H) = 0$; this gives us

$$r_H = \sqrt[p]{\mu}. \quad (\text{A.3})$$

We can rewrite the function $f_p(r)$ in terms of the horizon:

$$f_p(r) = \frac{r^2}{L^2} \left(1 - \left(\frac{r_H}{r} \right)^p \right). \quad (\text{A.4})$$

We first make the transformation to Euclidean time, $t \rightarrow -i\tau$:

$$ds^2 = f(r)d\tau^2 + f(r)^{-1}dr^2 + r^2d\Omega_2^2. \quad (\text{A.5})$$

Lets separate out the angular dependence, considering only the manifold of τ and r . Then we have a metric described by

$$dz^2 = f(r)d\tau^2 + f(r)^{-1}dr^2. \quad (\text{A.6})$$

Suppose we consider r just outside the horizon: $r = r_H + \epsilon$. We will determine the approximate spacetime metric, keeping terms of order 1 and ϵ . The functional form of $f(r)$ initially looks like (without throwing out terms)

$$\begin{aligned} f(r_H + \epsilon) &= \frac{(r_H + \epsilon)^2}{L^2} \left(1 - \left(\frac{r_H^p}{(r_H + \epsilon)^p} \right) \right) \\ &= \frac{1}{L^2} \left(r_H^2 + 2r_H\epsilon + \epsilon^2 - \frac{r_H^p(r_H + \epsilon)^2}{(r_H + \epsilon)^p} \right). \end{aligned} \quad (\text{A.7})$$

We will now make the Taylor series approximation (at $\epsilon = 0$) $(r_H + \epsilon)^{2-p} \simeq r_H^{2-p}(1 - (2-p)\frac{\epsilon}{r_H})$. We have only kept terms of the approximation to $\mathcal{O}(\epsilon)$.

$$\begin{aligned} f(r_H + \epsilon) &\simeq \frac{1}{L^2} \left(r_H^2 + 2r_H\epsilon + \epsilon^2 - r_H^2 \left(1 - (2-p)\frac{\epsilon}{r_H} \right) \right) \\ &= \frac{1}{L^2} (r_H^2 + 2r_H\epsilon + \epsilon^2 - r_H^2 + (2-p)r_H\epsilon) \\ &= \frac{1}{L^2} (pr_H\epsilon + \epsilon^2). \end{aligned} \quad (\text{A.8})$$

Finally we toss out the ϵ^2 term to obtain the final $\mathcal{O}(\epsilon)$ approximation:

$$f(r_H + \epsilon) \simeq \frac{pr_H\epsilon}{L^2}. \quad (\text{A.9})$$

This gives us a spacetime metric for our manifold

$$dz^2 = \frac{pr_H \epsilon}{L^2} d\tau^2 + \frac{L^2}{pr_H \epsilon} d\epsilon^2. \quad (\text{A.10})$$

We will now make an appropriate coordinate transformation to put the metric in suggestive form. First, let $\xi = 2\sqrt{\frac{\epsilon}{pr_H}}L$. Then $\xi^2 = \frac{4\epsilon L^2}{pr_H}$ and $d\xi^2 = \frac{L^2 d\epsilon^2}{pr_H \epsilon}$. After substitution we obtain

$$dz^2 = \xi^2 \frac{p^2 r_H^2}{4L^4} d\tau^2 + d\xi^2. \quad (\text{A.11})$$

We will now make another coordinate transformation, $\chi = \frac{pr_H \tau}{2L^2}$. Then $d\chi^2 = \frac{p^2 r_H^2}{4L^4} d\tau^2$. This gives us the metric

$$dz^2 = \xi^2 d\chi^2 + d\xi^2. \quad (\text{A.12})$$

This geometry is, essentially polar coordinates in spacetime with one spatial dimension and one (Euclidean) time dimension. There is an issue, however, with the unrestricted χ variable, which is timelike; without restrictions to make χ periodic in time, we obtain a conical singularity and our spacetime becomes geodesically incomplete. Fortunately we are free to restrict χ in this sense. For constant ξ , which correspond to circles in Euclidean time under our χ transformation, we want

$$\oint dz = 2\pi\xi. \quad (\text{A.13})$$

for constant ξ we have $dz = \xi d\chi$, so we want to restrict $\chi \in [0, 2\pi]$. Hence, we restrict τ to be in $[0, \beta]$, where

$$\beta = \frac{4\pi L^2}{pr_H}. \quad (\text{A.14})$$

This β gives us the periodicity in Euclidean time, and produces a geodesically complete spacetime solution. From the rules of AdS/CFT, we identify this periodicity as the inverse

of the Hawking temperature of the black hole [11] [12]. That is, we identify the partition function given by

$$Z = \text{Tr} e^{-\beta\mathcal{H}}, \quad (\text{A.15})$$

(where \mathcal{H} is the Hamiltonian) with

$$e^{-i\mathcal{H}\Delta t}, \quad (\text{A.16})$$

where Δt is the periodic time interval. (Now it becomes apparent why we made the transformation to Euclidean time, besides making the form of A.12 suggestive.) So, we have

$$T_{Hawking}(r_H) = \frac{p r_H}{4\pi L^2}. \quad (\text{A.17})$$

This procedure can be applied to even more general $f(r)$ solutions.

Bibliography

- [1] Tinkham, Michael, *Introduction to Superconductivity, Second Edition*, Dover Publications, 2004.
- [2] Hartnoll, Sean A., Christopher P. Herzog, and Gary T. Horowitz, *Building an AdS/CFT Superconductor*, arXiv:0803.3295v1 [hep-th], 22 March 2008.
- [3] Albrecht, Dylan, Christopher C. Carone, and Joshua Erlich, *Deconstructing Superconductivity*, arXiv:submit/0532081 [hep-th], 13 August 2012.
- [4] Hartnoll, Sean. A., Christopher P. Herzog, and Gary T. Horowitz, *Holographic Superconductors*, arXiv:0810.1563v1 [hep-th], 9 October 2008.
- [5] Maldacena, Juan, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2**, 231 (1998) arXiv:hep-th/9711200.
- [6] Bousso, Raphael, *The Holographic Principle*, arXiv:hep-th/0203101v2, 29 June 2002.
- [7] Sakurai, Jun John, *Modern Quantum Mechanics (Revised Edition)*, Addison Wesley Publishing Company, 1994.
- [8] Johnson, Clifford V., *D-Branes*, Cambridge Monographs on Mathematical Physics, 2003.

- [9] McGreevy, John, 8.821 String Theory, Fall 2008. (Massachusetts Institute of Technology: MIT OpenCourseWare), <http://ocw.mit.edu> (Accessed 4/9/2013). License: Creative Commons BY-NC-SA.
- [10] Siopsis, George, *Lecture Notes on AdS/CFT Correspondence*, Fall 2010, <http://aesop.phys.utk.edu/ads-cft/notes.pdf>.
- [11] Zwiebach, Barton, *A First Course in String Theory, Second Edition*, Cambridge University Press, 2009.
- [12] Kapusta, Joseph I., and Charles Gale, *Finite Temperature Field Theory and Applications, Second Edition*, Cambridge Monographs on Mathematical Physics, 2006.