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Braid Groups and Evolution Algebras

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Braid Groups and Evolution Algebras

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Abstract

In this paper, we explore the relationship between the braid group and evolution algebras. First, we explore the braid group and how it is constructed. Next we discuss the Burau representation, and its relationship with free differential calculus and Magnus representations. Finally we use the Burau representation to create a new representation from the braid group directly to evolution algebras.

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Chapter 1

Background Information

1.1 Braids

To begin, we must explore the concept of a braid and the Braid group. The most natural way to define a braid is by using geometry. Consider the points $p_i = (i, 0, 0)$ and $q_i = (i, 0, 1) \in \mathbb{R}^3$, where $i = 1, 2, 3, ..., n$. Now join each p_i to some q_j with a polygonal arc. Each p_i must be joined to a q_j but i does not need to equal j. No p_i may be joined to another p_j, and likewise for the q_i's. Each individual arc is considered a strand (sometimes called a string). For this object to be considered a braid on n strands or a n-braid, each horizontal plane that cuts through the strings must intersect each string only once. For examples see figures 1.1 and 1.2.

Although braids as geometrically described are three dimensional objects, they can be represented by two dimensional projections. While these projections lose some properties that comes in three dimensions, they still denote when two strands are crossed and maintain which strand crosses on top of the other. As an example, figure 1.3 shows the 2 dimensional projection of the braid in figure 1.2.

Two n-braids are defined to be equivalent if a braid can be transformed into another using elementary knot moves, i.e. no strands are untangled, only shifted up, down, left or

Figure 1.1: Example of a not a braid

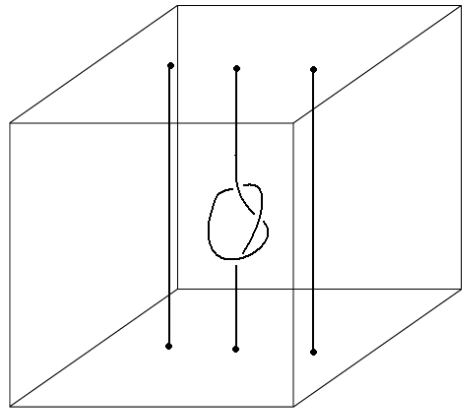


Figure 1.2: Example of a 3-braid

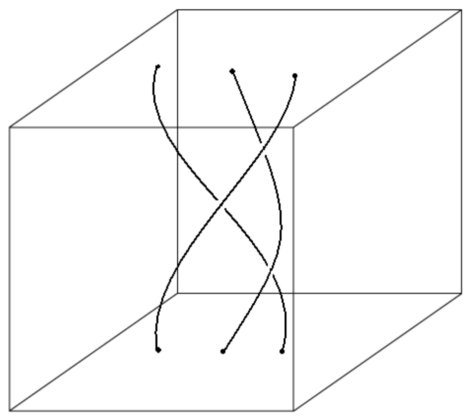
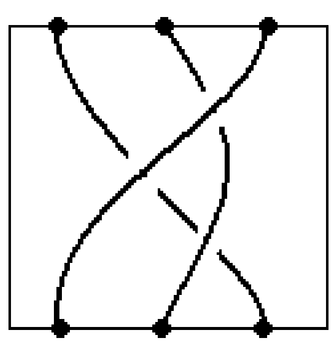


Figure 1.3: Example of 2 dimensional projection



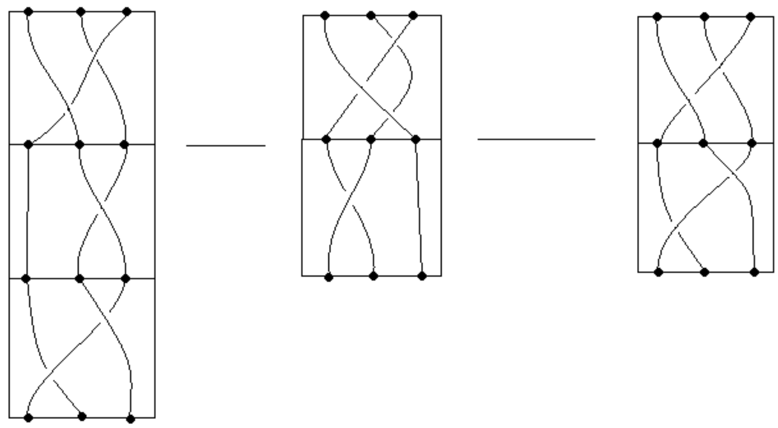
right, then these two braids are equivalent. (Note that all of these moves are reversible.) In addition, these deformations cannot cause any of the strands to intersect each other more than they did previously, so in the end the braids will still have the same number of crosses and the same strands will cross. Let's verify that this is an equivalence relation. Let α, β, γ be n-braids. Then α is equivalent to itself. Now if α is equivalent to β, α can be continuously deformed into β, and if we reverse these moves and use them on β we will produce α. Therefore β is also equivalent to α. Finally if α is equivalent to β and β is equivalent to γ we can deform α into β and continue to deform β into γ. Then we have deformed α all the way to γ making them equivalent. Now we have an equivalence relation for braids so braids can be categorized into equivalence classes. These equivalence classes are denoted by one representative from the class and are also referred to as braids, nevertheless the meaning will be evident with context. The set of all equivalence classes will become the braid group, once we define the appropriate binary operation on this set.

Braid multiplication is defined by placing braids end to end. For example, take braids α and β, then $\alpha\beta$ is the union of α contracted to half its height and β contracted to half its height. We can also think of this as the object where α joins the points p_i to the point q_i. Now we can write a third set of points r_i's where $r_i = (i, 0, 2)$ and we can join the q_i's to the r_i's by β.

So $\alpha\beta$ is the braid which joins the p_i's to the r_i's. Thus we get a braid which is twice as long, but, again, it can be contracted to the original height. This operation is associative, i.e. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, since $\alpha\beta$ will be contract in to half its height (thus α and β are a fourth their original height) when multiplied with γ. Next we can take $(\alpha\beta)\gamma$ and contract the γ section to $\frac{1}{4}$ its original height and stretch the α section of the braid to $\frac{1}{2}$ its original height. Finally we have the braid $\alpha(\beta\gamma)$. Thus this operation is associative. This can also be visually verified by figure 1.4.

We must also verify that braid multiplication is well-defined on equivalence classes. Let α and β be equivalent braids. We must show that $\alpha\gamma = \beta\gamma$, for each γ. We know

Figure 1.4: Example of braid associativity



that α can be continuously deformed into β. So if we take the braid $\alpha\gamma$ can consider it to be the braid where α joins the points p_i to the points q_i and then γ joins the points q_i to the points r_i as we have seen before. We can deform the α section for the braid into β and we will get the braid $\beta\gamma$. Thus the two braids are equivalent and the operation is well defined. Now that we have a binary operation, which is well defined on the equivalence classes of braids, we can define the braid group, denoted $\mathbf{B_n}$, as the set of all equivalence classes of n-braids with braid multiplication as the binary operation. The identity element in $\mathbf{B_n}$ is the braid of n parallel strands, i.e., each p_i is jointed to q_i without any crosses. This braid is also called the trivial braid. Inverses are also easy to determine. If β is a braid then β^{-1} is the mirror image in the plane $z = \frac{1}{2}$. Thus $\mathbf{B_n}$ is indeed a group.

Now that we have established the more intuitive geometric definition, let's explore another way to define the braid group, using generators and relations. First, we will establish the generators and relations in $\mathbf{B_n}$ and give an example in a small n. In order to do this we must note that two braids are still equivalent if we change the altitude of a cross. For example the two braids in figure 1.5 are clearly equivalent despite the change in altitude of the crosses.

So all braids can be rewritten such that all crosses occur at different heights. Now we

Figure 1.5: Example of Altitude changes

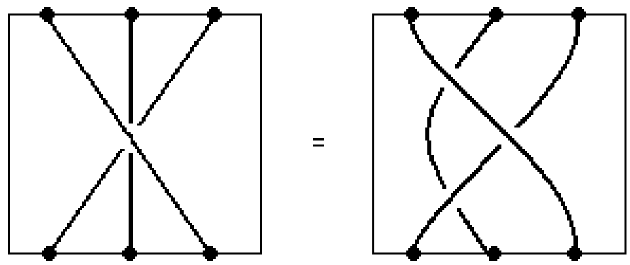
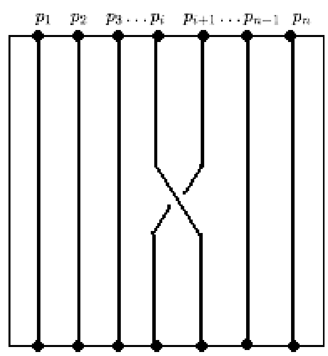


Figure 1.6: σ_i



can cut the braid into smaller braids that only have one cross of adjacent strands. Each of these individual crosses can be considered a generator. Thus the generator σ_i is the braid which consists of the i strand crossing over the $i+1$ strand. This generator has the inverse σ_i^{-1} which is the braid which has the i strand cross under the $i+1$ strand. (See figures 1.6 and 1.7 for examples.)

Consequently $\mathbf{B_n}$ has $n-1$ generators. Now we can write any braid, β, as the product of various σ_i and σ_i^{-1}, for some $i, \in \{1, 2, \ldots, n-1\}$. But we know that some braids are equivalent even though they have different crosses at different places. In order to solve this problem we must define relations between the generators, which allow us to use our previous transformations.

These relations come in two distinct varieties:

Figure 1.7: $\sigma_1,\ \sigma_2,\ \sigma_3$ in $\mathbf{B_n}$

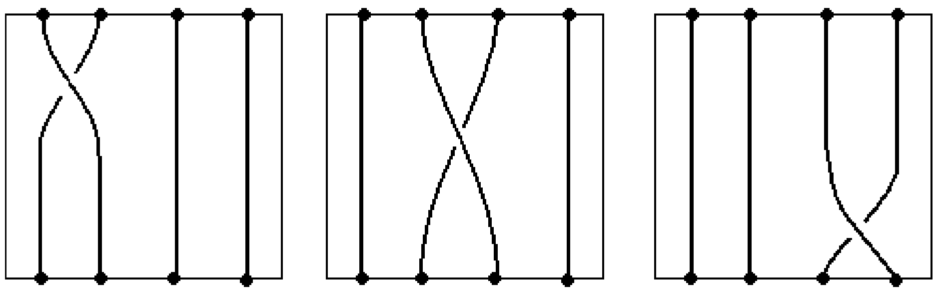
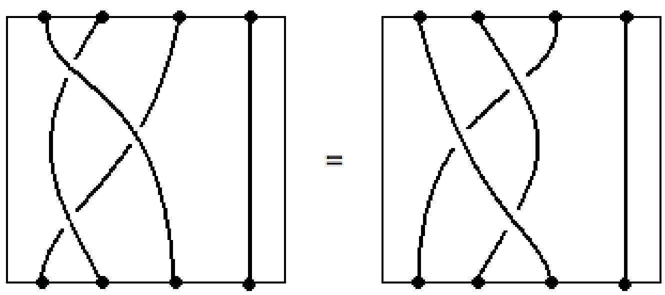


Figure 1.8: $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$



1. $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$

2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \cdots, n-2$

These relations are often referred to as the *braid relations* or as the *Artin relations*. They can be easily seen in the case $n = 4$. In this case there are 3 generators $(\sigma_1, \sigma_2, \sigma_3)$ and 3 relations:

1. $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$,

2. $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$,

3. $\sigma_1\sigma_3 = \sigma_3\sigma_1$.

It is easiest to verify these relations on $\mathbf{B_4}$ visually through figures 1.8 and 1.9.

Figure 1.9: $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$

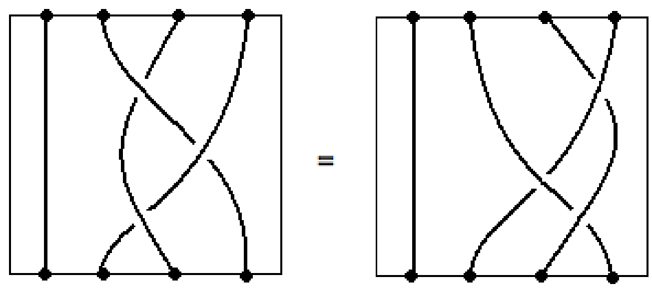
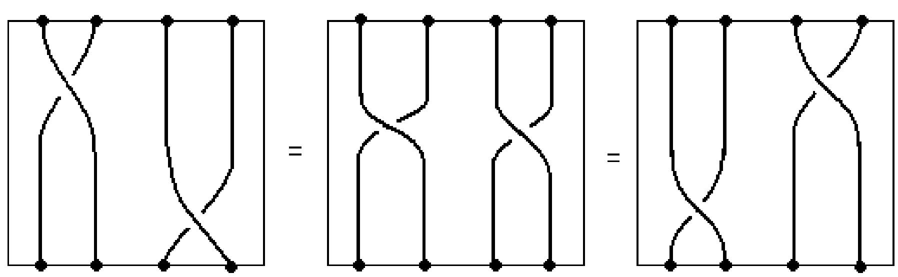


Figure 1.10: $\sigma_1\sigma_3 = \sigma_3\sigma_1$



These relations can be used to reduce a braid from its series of generators to a simpler braid, which will represent the equivalence class. There is an algorithm the *Duhornoy algorithm*, which can be used to systematically reduce braids, written in terms of their generators, into their simplest form, and thus into their equivalence classes. For more information on this algorithm see [Richard H. Crowell, Ralph H. Fox, Introduction to Knot Theory, 1963].

1.2 Free Groups, Presentations and the Braid Group

In order to understand and apply this definition of the braid group we need to quickly discuss free groups and presentations. A group, G, is considered free if there is a set $\{g_1, g_2, ..., g_n\} \subset G$ such that all elements of G can be written as a finite product of these elements and their inverses, without any defining relations. We often denote this set of generators, S and the Free group on S as $F(S)$. The elements of a free group are often called words, when they are written as a string of generators. Two of these words are multiplied by putting them end to end. The neutral element of the group is the word with no generators. The inverse of an element can be written as the inverse of each generator (where $(g_i^{-1})^{-1} = g_i$) written in the reverse order. For example take the word $g_1 g_3 g_2^{-1}$; the inverse element is $g_2 g_3^{-1} g_1^{-1}$. One important theorem about free groups is as follows:

Theorem 1.1. *Let G be a a free group generated by $S = \{s_i | i \in I\}$ and let G' be any other group. If s_i' for $i \in I$ are any elements G', not necessarily distinct, then there is exactly one homomorphism $\phi : G \to G'$ such that $\phi(s_i) = s_i'$.*

This theorem allows us to define a function just on the generators of a free group, and know that it will induce a function on the whole group. This works since free groups have no relations (as the braid group does). Therefore in order to define and work with a group which has relations, we use a system called a presentation.

A presentation of a group, G, denoted $< S|R >$, is a set of generators of G, called S

and a set of relations of G, called R, which can be used to define the group. Relations, in this sense, often called relators, are words, which is an element of the free group on S, that is defined as being equal to the identity element, 1. For the braid group the relations would be the words of the form $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$ for $|i-j| \geq 2$ and $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ for $i = 1, 2, \cdots, n-2$. We take this set of relations, which is a subset of the free group on S, and create the smallest normal subgroup containing R. Then $F(S)/N$ is isomorphic to G. So now we know that the braid group is isomorphic to the free group on n generators mod N, where N is the smallest normal subgroup containing the aforementioned relators.

1.3 Free Differential Calculus

Free Differential Calculus is a method to study groups defined by generators and relations. This tool allows us explore Magnus Representations, a generic type of representation for free group, from which we derive the Burau Representation of braids. First, for any group G, we must define the group ring, JG, associated with the ring J of integers. An element of this ring is a sum $\sum a_g g$ over all $g \in G$, where a_g is an integer and is equal to zero for all but a finite number of g. Thus $\sum a_g g$ is a finite sum. Addition is defined by $\sum a_g g + \sum b_g g = \sum (a_g + b_g) g$. Since only a finite number of a_g's and b_g's are nonzero only a finite number of $(a_g + b_g)$'s are non-zero so $\sum (a_g + b_g) g \in JG$. Multiplication is defined by $(\sum a_g g)(\sum b_g g) = \sum_g (\sum_h a_{gh^{-1}} b_h) g$. For the same reasoning, $\sum (\sum_h a_{gh^{-1}} b_h) g \in JG$.

An important homomorphism is the map $o : JG \to J$, which is induced by the trivial homomorphism $o : G \to 1$. Thus an element $\sum a_g g \in JG$ is mapped to the sum of its coefficients, a_g, since $o(\sum a_g g) = \sum a_g o(g) = \sum a_g$. This function is often called the trivializer. This homomorphism will be denoted, v^t. Now the kernel of the ring-homomorphism o, denoted $\mathcal{O}$, is the set of all elements in G which have coefficient sum zero. This ideal of JG is called the *fundamental ideal* of JG. This homomorphism will be used later.

Next we will define a derivative on the group ring JG. A derivative is any mapping $D : JG \to JG$ which satisfies

$$D(u + v) = D(u) + D(v), \tag{1.1}$$

$$D(u \cdot v) = D(u)v^t + u \cdot D(v) \tag{1.2}$$

such that $u, v \in JG$. Some important consequences are as follows:

1. $D(u \cdot v) = D(u)o(v) + u \cdot D(v)$ can be simplified to $D(gh) = D(g) + gD(h)$, for $g, h \in G$
2. $D(a) = 0$ for all $a \in JX$, where $a \in J$,
3. $D(\sum a_g g) = \sum a_g D(g)$,
4. $D(g^{-1}) = -g^{-1}D(g)$ for all $g \in G$.

Now let's examine what happens with these derivatives when the group is a free group defined by generators and relations. Let X be a free group with the set of generators, $(x) = \{x_1, x_2, \ldots\}$, which can be either finite of infinite. An element of X is the unique reduced word $\prod_{k=1}^{l} x_{j_i}^{\epsilon_i}$, where $\epsilon_k = 1 \; or -1, \epsilon_k + \epsilon_{k+1} \neq 0$ if$j_k = j_{k+1}$. Now an element of the free group ring JX is a *free polynomial* $f(x) = \sum a_u u$, where $u \in X$ and $a_u \in J$, where all but a finite number of the $a_u = 0$. Let's examine a homomorphism ϕ from X to a group G which maps $(x) = \{x_1, x_2, \ldots\}$ into $\phi((x)) = \{\phi(x_1), \phi(x_2), \ldots, \phi(x_n)\}$. This induces a ring homomorphism $\phi' : JX \to JG$ defined by $\phi'(f(x)) = \sum a_u \phi(u)$, where $f(x) \in JX$. Again we can look at the the ring homomorphism $o : JX \to J$ define by $o(f(x)) = \sum a_u o(x) = \sum a_u * 1 = \sum a_u$. This can be considered the coefficient sum of $f(x)$ and is often denoted $f(1)$. The fundamental ideal $\mathcal{X}$, based on this ring homomorphism, is the set of all free polynomials such that the coefficient sum is zero. Now we can examine derivatives on a free group. For each generator, x_i of X, we can define a derivative, $\partial f(x)/\partial x_i$, which has the property:

$$\frac{\partial x_k}{\partial x_i} = \delta_{i,k} \text{ (Kroneker delta)} \tag{1.3}$$

In addition, there is one and only one derivation $f(x) \rightarrow f'(x)$ mapping $x_1, x_2, ...$ to prescribed elements $h_1(x), h_2(x), ... \in JX$ by

$$f'(x) = \sum \frac{\partial f(x)}{\partial x_i} \cdot h_j(x) \tag{1.4}$$

We must verify that these definitions actually are derivatives as previously defined. First for each index i and element $u \in X$ define:

$$\begin{aligned} < i, u > &= 1 \text{ if } x_i \text{ is an initial segment of the reduced word } u \text{, and} \\ < i, u > &= 0 \text{ otherwise.} \end{aligned}$$

Next we extend this definition to all of JX:

$$\begin{aligned} < i, f(x) > &= < i, \sum a_u u > \\ &= \sum a_u < i, u > . \end{aligned}$$

Next we define for any $i \in \mathbb{Z}$, $w \in X$ and $f(x) \in JX$:

$$< i, w, f(x) > \;=\; < i, w^{-1} f(x) > - < i, w - 1 > f(1).$$

Then $< i, w, u > \;=\; 0$ when w is not an initial segment of u. Also we see that $< i, w, f(x) > \;=\; < i, w, \sum a_u u > \;=\; \sum a_u < j, w, u >$ is equal to zero for all but a finite number of $w \in X$. So we can define $\frac{\partial f(x)}{\partial x_i}$ as $\sum_{w \in X} < j, w, f(x) > w$. It is easy to see that the condition $D(u+v) = D(u) + D(v)$ for all $f(x), g(x) \in JX$, since

$$\begin{aligned}
\frac{\partial(f(x)+g(x))}{\partial x_i} &= \sum_{w \in X} < j, w, f(x)+g(x) > w \\
&= < i, w^{-1}(f(x)+g(x)) > - < i, w-1 > (f(1)+g(1)) \\
&= < i, w^{-1}f(x) + w^{-1}g(x) > - < i, w-1 > f(1) - < i, w-1 > g(1) \\
&= < i, w^{-1}f(x) > + < i, w^{-1}g(x) > - < i, w-1 > f(1) - < i, w-1 > g(1) \\
&= < i, w^{-1}f(x) > - < i, w-1 > f(1) + < i, w^{-1}g(x) >< i, w-1 > g(1) \\
&= < i, w, f(x) > + < i, w, g(x) > \\
&= \frac{\partial f(x)}{\partial x_i} + \frac{\partial g(x)}{\partial x_i}
\end{aligned}$$

We can also verify the simplified version of 1.2. Let $u, v \in X$ then

$$\begin{aligned}
\frac{\partial(uv)}{\partial x_i} &= \sum_{w \in X} < j, w, uv > w \\
&= \sum_{w} (< i, w^{-1}uv > - < i, w^{-1} >) w \\
&= \sum_{w} (< i, w^{-1}u > - < i, w^{-1} >) w + \sum_{w} (< i, w^{-1}uv > - < i, w^{-1}u >) w \\
&= \sum_{w} (< i, w^{-1}u > - < i, w^{-1} >) w + u \sum_{w} (< i, t^{-1}v > - < i, t^{-1} >) t \\
&= \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}
\end{aligned}$$

Now let us examine equation 1.3. Note that the only initial segments of x_k are 1 and x_k so that:

$$\begin{aligned}
\frac{\partial x_k}{x_i} &= < i, 1, x_k > + < j, x_k, x_x > x_k \\
&= (< i, x_k > - < i, 1 >) + (< i, 1 > - < i, x_k^{-1} >) x_k \\
&= (\delta_{ik} - 0) + (0 - 0) x_k \\
&= \delta_{ik}.
\end{aligned}$$

To continue, we need to finish the proof of equation 1.4. We know $\sum_i \frac{\partial f(x)}{\partial x_i} \cdot h_i(x)$ is finite, since $\frac{\partial f(x)}{\partial x_i} = 0$ for all but a finite number of indices. The map which sends $f(x)$ to

$\sum_i \frac{\partial f(x)}{\partial x_i} \cdot h_i(x)$, is a derivation since

$$\begin{aligned}\sum_i \frac{\partial f(x)}{\partial x_i} \cdot h_i(x) + \sum_i \frac{\partial g(x)}{\partial x_i} \cdot h_i(x) &= \sum_i \frac{\partial f(x)}{\partial x_i} \cdot h_i(x) + \frac{\partial f(x)}{\partial x_i} \cdot h_i(x) \\ &= \sum_i (\frac{\partial f(x)}{\partial x_i} + \frac{\partial f(x)}{\partial x_i}) \cdot h_i(x) \\ &= \sum_i \frac{\partial f(x) + g(x)}{\partial x_i} \cdot h_i(x)\end{aligned}$$

So addition is preserved. Now we need to show this definition respects the multiplication rule.

$$\begin{aligned}\sum_i \frac{\partial f(x)g(x)}{\partial x_i} \cdot h_i(x) &= \sum_i \left[\frac{\partial f(x)}{\partial x_i} g(1) + f(x)\frac{\partial g(x)}{\partial x_i}\right] \cdot h_i(x) \\ &= \sum_i \left[\frac{\partial f(x)}{\partial x_i} g(1) \cdot h_i(x) + f(x)\frac{\partial g(x)}{\partial x_i} \cdot h_i(x)\right] \\ &= \sum_i \frac{\partial f(x)}{\partial x_i} g(1) \cdot h_i(x) + \sum_i f(x)\frac{\partial g(x)}{\partial x_i} \cdot h_i(x) \\ &= g(1)\sum_i \frac{\partial f(x)}{\partial x_i} \cdot h_i(x) + f(x)\sum_i \frac{\partial g(x)}{\partial x_i} \cdot h_i(x)\end{aligned}$$

So $\sum_i \frac{\partial f(x)}{\partial x_i} \cdot h_i(x)$ truly is a derivative. Next we can see x_k is mapped to h_k, by this derivative. Now we must show uniqueness. So if $f(x) \to f'(x)$ is a derivative mapping which maps $x_i \to h_i$ then $f(x) \to f'(x) - \sum_i \frac{\partial f(x)}{\partial x_i} \cdot h_i(x)$ is a derivation map which maps each x_i to zero and every x_i^{-1} to zero. Thus all elements of JX are mapped to 0. So $f'(x) = \sum_i \frac{\partial f(x)}{\partial x_i} \cdot h_i(x)$. Now we've show that (1.4) is in fact a correct definition of derivatives. Due to all of this, we know we can find a derivative for all $f(x) \in JX$. We can also define a "chain rule"; let λ is a homomorphism of a free group Y, with generations y_i into a free group X, with generators x_j, then for any $f \in JY$, $\frac{\partial \lambda(f)}{\partial x_j} = \sum_i \lambda\left(\frac{\partial f}{\partial y_i}\right)\frac{\partial \lambda(y_i)}{\partial x_j}$. This will be more important later. If you wish to know more about the specifics of finding these derivatives see [Fox, Ralph H (1952)].

Now we can inductively define higher order derivatives recursively:

$$\frac{\partial^n f(x)}{\partial x_{i_n} \partial x_{i_{n-1}} ... \partial x_{i_1}} = \frac{\partial}{\partial x_{i_n}} (\frac{\partial^{n-1} f(x)}{\partial x_{i_{n-1}} ... \partial x_{i_1}}) \tag{1.5}$$

This can also be written $D_{i_1 i_2 ... i_n}$. We should also note that $D_{i_1 i_2 ... i_n}(f(x)g(x)) = \sum_{p=1}^{n} D_{i_i i_2 ... i_n} f(x) D_{i_i i_2 ... i_n} g(1) + f(x) D_{i_i i_2 ... i_n} g(x)$ by induction from equation (1.2).

1.4 Magnus Representations

Using these derivatives we can define Magnus representations. Let $\overline{S_n}$ be a free Abelian semi-group with generators $\overline{s_1}, ..., \overline{s_n}$. Let $\mathcal{R}$ be a ring and let $A_0(\mathcal{R}, \overline{S_n})$ be the semi-group ring of $\overline{S_n}$ with respect to $\mathcal{R}$. So elements of $A_0(\mathcal{R}, \overline{S_n})$ are of the form $\sum_{i=1}^{n} r_i \overline{s_i}^{k_i}$, where $r_i \in \mathcal{R}$, $\overline{s_i}$ is a generator, and $k_i \in \mathbb{R}^+$. We can also consider $\mathbf{F_n}$, the free group on the generators $x_1, ..., x_n$. Now we can define a map $\tau : \mathbf{F_n} \to GL(2, A_0(J\mathbf{F_n}, \overline{S_n}))$ by the rule for all $w \in \mathbf{F_n}$, $\tau(w)$ is the matrix:

$$[w] = \begin{bmatrix} w & \sum_{j=1}^{n} \frac{\partial w}{\partial x_j} \overline{s_j} \\ 0 & 1 \end{bmatrix} \tag{1.6}$$

Therefore $\tau(x_i)$ is the matrix:

$$[x_i] = \begin{bmatrix} x_i & \overline{s_i} \\ 0 & 1 \end{bmatrix} \tag{1.7}$$

Let's show that $[wv] = [w][v]$. We already know

$$[wv] = \begin{bmatrix} wv & \sum_{j=1}^{n} \frac{\partial wv}{\partial x_j} \overline{s_j} \\ 0 & 1 \end{bmatrix}$$

and we also know that

$$\begin{aligned}\sum_{j=1}^{n}\frac{\partial wv}{\partial x_j}\overline{s_j} &= \sum_{j=1}^{n}(\frac{\partial w}{\partial x_j}+\frac{\partial v}{\partial x_j}w)\overline{s_j} \\ &= \sum_{j=1}^{n}\frac{\partial w}{\partial x_j}\overline{s_j}+\frac{\partial v}{\partial x_j}w\overline{s_j} \\ &= \sum_{j=1}^{n}\frac{\partial w}{\partial x_j}\overline{s_j}+\sum_{j=1}^{n}\frac{\partial v}{\partial x_j}w\overline{s_j} \\ &= \sum_{j=1}^{n}\frac{\partial w}{\partial x_j}\overline{s_j}+w\sum_{j=1}^{n}\frac{\partial v}{\partial x_j}\overline{s_j}\end{aligned}$$

thus

$$\begin{aligned}\begin{bmatrix} wv & \sum_{j=1}^{n}\frac{\partial wv}{\partial x_j}\overline{s_j} \\ 0 & 1\end{bmatrix} &= \begin{bmatrix} wv & \sum_{j=1}^{n}\frac{\partial w}{\partial x_j}\overline{s_j}+w\sum_{j=1}^{n}\frac{\partial v}{\partial x_j}\overline{s_j} \\ 0 & 1\end{bmatrix} \\ &= \begin{bmatrix} w & \sum_{j=1}^{n}\frac{\partial w}{\partial x_j}\overline{s_j} \\ 0 & 1\end{bmatrix}\begin{bmatrix} v & \sum_{j=1}^{n}\frac{\partial v}{\partial x_j}\overline{s_j} \\ 0 & 1\end{bmatrix} \\ &= [w][v]\end{aligned}$$

So, now we proved that the mapping τ is a representation. Let's use the symbol $[\mathbf{F_n}]$ for the image of $\mathbf{F_n}$ under τ. The mapping $\tau : \mathbf{F_n} \to [\mathbf{F_n}]$ is called the Magnus representation of $\mathbf{F_n}$. However, it is not particularly useful since $w \in \mathbf{F_n}$ is still used in $[w]$. So let ϕ be an homomorphism mapping $\mathbf{F_n}$ to another group. As we have shown before we can extend this homomorphism to a ring-homomorphism over $J\mathbf{F_n}$. Then we can define $\phi([w])$ as the matrix

$$\phi([w]) = \begin{bmatrix} \phi(w) & \sum_{j=1}^{n}\phi(\frac{\partial w}{\partial x_j}\overline{s_j}) \\ 0 & 1\end{bmatrix} \tag{1.8}$$

This matrix no longer has any elements of $\mathbf{F_n}$ in it but still remains a representation since all the properties above are maintained by a homomorphism. Now let $[\mathbf{F_n}]^{\phi}$ be the image of $[\mathbf{F_n}]$ under ϕ. This mapping $\phi : \mathbf{F_n} \to [\mathbf{F_n}]^{\phi}$ as defined before will be called the *Magnus ϕ-representation*.

Now we can generalize the Magnus ϕ-representation to a representation of $\mathbf{F_n}$ using $k \times k$ upper triangular matrices. We already know that higher order derivatives exist, but here we will define a specific derivative. Let

$$\begin{aligned} D(w) &= \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} \overline{s_i} \\ D^q(w) &= D(D^{q-1}(w)). \end{aligned}$$

Next, we let ϕ be a homomorphism acting on $\mathbf{F_n}$ and we induce the ring homomorphism ϕ on $J\mathbf{F_n}$. Let $\phi(D^q(w))$ be the image of $D^q(w)$, the derivative of w, under the ring-homomorphism. Then we can define for any $w \in \mathbf{F_n}$

$$\phi(\{w\}) = \begin{bmatrix} \phi(w) & \phi(D(w)) & \phi(D^2(w)) & \phi(D^3(w)) & \dots & \phi(D^{n-1}(w)) \\ 0 & 1 & D(w)^t & D^2(w)^t & \dots & D^{n-2}(w)^t \\ 0 & 0 & 1 & D(w)^t & \dots & D^{n-3}(w)^t \\ 0 & 0 & 0 & 1 & \dots & D^{n-4}(w)^t \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.9}$$

where $D(w)^t$ is the previously defined trivializer, which produces the coefficient sum. This new map is also a representation. We can verify this using the the product rule, as we did before for the Magnus representation. The proof will be omitted due to the size of the matrices. However the braid group is not a free group. Nevertheless it has a faithful representation as a subgroup of Aut $\mathbf{F_n}$, or the group of of right automorphisms on the free group $\mathbf{F_n}$, which will be proved in the next section. So we need to expand our idea of Magnus representations to subgroups of Aut $\mathbf{F_n}$. Let ϕ be any homomorphism acting on $\mathbf{F_n}$, as before. Let's take any arbitrary subgroup of Aut $\mathbf{F_n}$, call it A_ϕ, which satisfies the condition $\phi(x) = \phi(\alpha(x))$ for all $x \in \mathbf{F_n}$ and $\alpha \in A_\phi$. For all $\alpha \in A_\phi$ define $\|\alpha\|^\phi$ to be the $n \times n$ the matrix:

$$\|\alpha\|^\phi = \left[\phi\left(\tfrac{\partial(\alpha(x_i))}{\partial x_j}\right)\right] \tag{1.10}$$

with entries in $J\mathbf{F_n}^{\phi}$. The mapping $\tau : \alpha \rightarrow \|\alpha\|^{\phi}$ is the Magnus representation of A_{ϕ}. We must show that it is a group homomorphism from A_{ϕ} to the multiplicative group of $n \times n$ matrices over $J\mathbf{F_n}^{\phi}$ to verify it is a representation. Suppose $\alpha \in A_{\phi}$ is the identity automorphism. Then $\phi\Big(\frac{\partial \alpha(x_i)}{\partial x_j}\Big) = \phi\Big(\frac{\partial (x_i)}{\partial x_j}\Big) = \delta_{ij}$. So $\|\alpha\|^{\phi}$ is the identity matrix, since $\delta_{ij} = 1$ only when $i = j$, which corresponds to the diagonal. Now fix $\alpha, \beta \in A_{\phi}$ where $\alpha(x_i) = w_i(x_1, ..., x_n)$ and $\beta(x_i) = v_i(x_1, ..., x_n)$ with $w_i, v_i \in \mathbf{F_n}$. Now we have, by the chain rule,

$$\begin{aligned} \frac{\partial(\beta(\alpha(x_i))}{\partial x_j} &= \frac{\partial}{\partial x_j}(w_i(v_1, ..., v_n)) \\ &= \sum_{k=1}^{n} \Big(\frac{\partial w_i}{\partial v_k}\Big)\Big(\frac{\partial v_k}{\partial x_j}\Big) \end{aligned}$$

But we know $\phi(x) = \phi(\beta(x))$ for all $x \in \mathbf{F_n}$. So since $v_k = \beta(x_k)$ we know

$$\begin{aligned} \phi\Big(\frac{\partial w_i}{\partial v_k}\Big) &= \phi\Big(\frac{\partial w_i}{\partial x_k}\Big) \\ &= \phi\Big(\frac{\partial \alpha(x_i)}{\partial v_k}\Big) \end{aligned}$$

Therefore

$$\begin{aligned} \phi\Big(\frac{\partial(\beta(\alpha(x_i)}{\partial x_j}\Big) &= \sum_{k=1}^{n} \phi\Big(\frac{\partial w_i}{\partial x_k}\Big)\phi\Big(\frac{\partial v_k}{\partial x_j}\Big) \\ &= \sum_{k=1}^{n} \Big(\frac{\partial a(x_i)}{\partial x_k}\Big)\Big(\frac{\partial \beta(x_k)}{\partial x_j}\Big) \end{aligned}$$

Now, by the definition of matrix multiplication, we have $\|\alpha\beta\|^{\phi} = \|\alpha\|^{\phi}\|\beta\|^{\phi}$. Next we will apply this representation to the Braid group and get the Burau representation.

1.5 Burau representation

As noted before, $\mathbf{B_n}$ has a faithful representation as a subgroup of Aut $\mathbf{F_n}$, where the generators of $\mathbf{F_n}$ are $x_1, ..., x_n$. In this section, we prove that assertion. The representation is induced by the map $\xi : \mathbf{B_n} \rightarrow$ Aut $\mathbf{F_n}$ defined by:

$$\begin{aligned} \xi(\sigma_i) \quad &: x_i \rightarrow x_i x_{i+1} x_i^{-1} \\ & x_{i+1} \rightarrow x_i \\ & x_j \rightarrow x_j \text{ if } j \neq i, i+1. \end{aligned}$$

First we notice that $\xi(\sigma_i)$ is an automorphism, since ξ is 1-1 on the generators of $\mathbf{F_n}$ it is also 1-1 on all of $\mathbf{F_n}$. So to verify that it is a representation we must verify that the relations hold. First we must verify $\xi(\sigma_i)\xi(\sigma_{i+1})\xi(\sigma_i) = \xi(\sigma_{i+1})\xi(\sigma_i)\xi(\sigma_{i+1})$ We know $\xi(\sigma_i)\xi(\sigma_{i+1})$ is defined by

$$\begin{aligned} & x_i \rightarrow x_i x_{i+1} x_{i+2} x_{i+1^{-1}} x_i^{-1} \\ & x_{i+1} \rightarrow x_i \\ & x_{i+2} \rightarrow x_{i+1} \\ & x_j \rightarrow x_j \text{ if } j \neq i, i+1, i+2. \end{aligned}$$

and so $\xi(\sigma_i)\xi(\sigma_{i+1})\xi(\sigma_i)$ is defined by

$$\begin{aligned} & x_i \rightarrow x_i x_{i+1} x_{i+2} x_{i+1^{-1}} x_i^{-1} \\ & x_{i+1} \rightarrow x_i x_{i+1} x_i^{-1} \\ & x_{i+2} \rightarrow x_i \\ & x_j \rightarrow x_j \text{ if } j \neq i, i+1, i+2. \end{aligned}$$

but $\xi(\sigma_{i+1})\xi(\sigma_i)$ is defined by

$$\begin{aligned} & x_i \rightarrow x_i x_{i+1} x_i^{-1} \\ & x_{i+1} \rightarrow x_i x_{i+2} x_i^{-1} \\ & x_{i+2} \rightarrow x_i \\ & x_j \rightarrow x_j \text{ if } j \neq i, i+1, i+2. \end{aligned}$$

and so $\xi(\sigma_{i+1})\xi(\sigma_i)\xi(\sigma_{i+1})$ is given by

$$\begin{aligned} & x_i \rightarrow x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \\ & x_{i+1} \rightarrow x_i x_{i+1} x_i^{-1} \\ & x_{i+2} \rightarrow x_i \\ & x_j \rightarrow x_j \text{ if } j \neq i, i+1, i+2. \end{aligned}$$

Thus $\xi(\sigma_i)\xi(\sigma_{i+1})\xi(\sigma_i) = \xi(\sigma_{i+1})\xi(\sigma_i)\xi(\sigma_{i+1})$. And thus, ξ respects the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. Our next step is to check that the relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$ is preserved. In order to do so we must show $\xi(\sigma_i)\xi(\sigma_j) = \xi(\sigma_j)\xi(\sigma_i)$ for $|i-j| \geq 2$. But we know that $\xi(\sigma_i)$ only moves x_i and x_{i+1} by since $|i-j| \geq 2$, $\xi(\sigma_j)$ will not move x_i or x_{i+1}. Likewise $\xi(\sigma_i)$ only moves x_j and x_{j+1}, which $\xi(\sigma_i)$ does not move. In both way you compose the functions you get the function defined by:

$$x_i \to x_i x_{i+1} x_i^{-1}$$
$$x_{i+1} \to x_i$$
$$x_j \to x_j x_{j+1} x_j^{1}$$
$$x_{j+1} \to x_j$$
$$x_k \text{ if } k \neq i, i+1, i+2.$$

Thus we have showed ξ is a representation map. Now we can think of $\mathbf{B_n}$ as a subgroup of Aut $\mathbf{F_n}$ and apply the Magnus Representation. Let $\mathbb{R}[t, t^{-1}]$ be the vector space, over $\mathbb{R}$, the integers, with the basis $\{\dots t^{-2}, t^{-1}, t, t^2, t^3, \dots\}$. Define a homomorphism $\psi : \mathbf{F_n} \to \mathbb{R}[t, t^{-1}]$ by $\psi(x_i) = t$ for all x_i. We can verify that $\psi(\xi(\sigma_i)(x_j)) = \psi(x_j)$ for all i, j. If $j \neq i$ or $i+1$ then $\xi(\sigma_i)(x_j) = x_j$. So $\psi(\xi(\sigma_i)(x_j)) = \psi(x_j)$. If $j = i$ then $\xi(\sigma_i)(x_j) = x_j x_{j+1} x_j^{-1}$, which implies $\psi(\xi(\sigma_i)(x_j)) = \psi(x_j x_{j+1} x_j^{-1}) = t \cdot t \cdot t^{-1} = t = \psi(x_j)$. If $j = i+1$ then $\xi(\sigma_i)(x_j) = x_i$ and in turn $\psi(\xi(\sigma_i)(x_j)) = \psi(x_i) = t = \psi(x_j)$. In all cases $\psi(\xi(\sigma_i)(x_j)) = \psi(x_j)$. Now we can write the Magnus representation of $\mathbf{B_n}$ over $\mathbb{R}[t, t^{-1}]$ where

$$\|\sigma_i\|^{\psi} = \left[\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \tag{1.11}$$

Since we've already proved Magnus representations are representations we know that this example is a representation. This representation map ρ, from $\mathbf{B_n}$ to the $n \times n$ matrices over $\mathbb{R}[t, t^{-1}]$, is called the *Burau representation.* One important quality for a representation is its faithfulness. A representation is said to be faithful if the only element it maps to 0 is the 0 element itself. While is seems like faithfulness is a good thing, it is often more helpful for a representation to not be faithful since this gives us another interesting aspect to study. It has been proved that the Burau representation is faithful for $n = 3$ and it has also been proved that the Burau representation is not faithful for for $n \geq 5$.[1] The faithfulness of the representation for $n = 4$ is still an open question. If a matrix $M = \|\beta\|^{\psi}$ for some $\beta \in \mathbf{B_n}$, we say that M is a a Burau matrix. Since this is a representation, we know that the set of all $n \times n$ Burau matrices, is closed under multiplication, has an identity element, and contains only invertible matrices and therefore this set is a group, denoted $\mathbf{M_n^B}$.

1.6 Evolution Algebras

We must also understand the concept of an *Evolution Algebra*. Any algebra can be defined by establishing generators and relations, so we must define the specific generators and relations for an evolution algebra. Let (A, $\cdot$) be an algebra over a field F such that if it admits a countable algebra basis $x_1, x_2, ..., x_n, \ldots$, such that

1. $x_i \cdot x_j = 0$, if $i \neq j$
2. $x_i \cdot x_i = \sum_k a_{i,k} x_k$, where $a_{i,k} \in F$, for any i.

Then we can call this algebra an evolution algebra. This basis is a set of generators, which can be combined in a linear combination with coefficients in the field, F. To define a specific algebra we must state what field it is over, how many generators are in the basis, and we must define what x_i^2 equals.

[1]For more information on the faithfulness of the Burau representation see S. Bigelow, *The Burau representation of the braid group* $\mathbf{B_n}$ *is not faithful for* $n = 5$, Geometry and Topology 3 1999 and J. Birman, *Braids, Links, and Mapping Class Groups*, 1975.

1.7 Products of Evolution Algebras

Now, we can define a specific type of multiplication on these evolution algebras as follows. Let A and B be evolution algebras on n generators over a field F. Let A be defined by the generators, $e_1, ..., e_n$, where

$$\begin{aligned} e_1^2 &= p_{11}e_1 + p_{12}e_2 + ... + p_{1n}e_n, \\ e_2^2 &= p_{21}e_1 + p_{22}e_2 + ... + p_{2n}e_n, \\ &\vdots \\ e_n^2 &= p_{n1}e_1 + p_{n2}e_2 + ... + p_{nn}e_n \end{aligned}$$

where $p_{ij} \in F$, and B be defined by the generators, $g_1, ..., g_n$, where

$$\begin{aligned} g_1^2 &= q_{11}g_1 + q_{12}g_2 + ... + q_{1n}g_n \\ g_2^2 &= q_{21}g_1 + q_{22}g_2 + ... + p_{2n}g_n \\ &\vdots \\ g_n^2 &= q_{n1}g_1 + q_{n2}g_2 + ... + q_{nn}g_n \end{aligned}$$

where $q_{ij} \in F$. Define $A * B$ as the evolution algebra over the field F, on n generators, $h_1, ..., h_n$, where

$$\begin{aligned} h_1^2 &= (\sum_k p_{1k}q_{k1})h_1 + (\sum_k p_{1k}q_{k2})h_2 + ... + (\sum_k p_{1k}q_{kn})h_n \\ h_2^2 &= (\sum_k p_{2k}q_{k1})h_1 + (\sum_k p_{2k}q_{k2})h_2 + ... + (\sum_k p_{2k}q_{kn})h_n \\ &\vdots \\ h_n^2 &= (\sum_k p_{nk}q_{k1})h_1 + (\sum_k p_{nk}q_{k2})h_2 + ... + (\sum_k p_{nk}q_{kn})h_n \end{aligned}$$

This multiplication is similar to matrix multiplication. In fact, it is possible to write A as the $n \times n$ matrix

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

and B is the same manner with the q_{ij}'s as the entries. Then these matrices can be multiplied and the product of the matrices can be converted back to an evolution algebra. This algebra is $A * B$.

Chapter 2

The Evolution Algebra Representation

2.1 Introduction to the Representation

By using the Burau representation, we can construct a representation of the braid group which sends braids to evolution algebras, instead of matrices. We begin by defining an evolution algebra based on a Burau matrix and showing that the set of all evolution algebras defined from a Burau matrices, with the previously defined multiplication, is a group. From this, we will define and prove a representation directly from the braid group to this group of evolution algebras.

2.2 Burau Matrices and Evolution Algebras

We must show that an Evolution algebra can be defined from a Burau matrix. Let $M = [m_{i,j}]$ be a $n \times n$ Burau matrix. Note that $m_{i,j} \in \mathbb{R}[t, t^{-1}]$ and $\mathbb{R}[t, t^{-1}]$ is an integral domain so it sits inside a field. So we can define evolution algebras with coefficients in this integral domain without any problems. Define an evolution algebra on n generators,

$e_1, e_2, \dots, e_n$, over $\mathbb{R}[t, t^{-1}]$ by

$$e_1^2 = m_{1,1}e_1 + m_{1,2}e_2 + ... + m_{1,n}e_n,$$
$$e_2^2 = m_{2,1}e_1 + m_{2,2}e_2 + ... + m_{2,n}e_n,$$
$$\vdots$$
$$e_n^2 = m_{n,1}e_1 + m_{n,2}e_2 + ... + m_{n,n}e_n$$

Because of the definition of algebra multiplication, we know that If $M = [m_{i,j}]$ and $P = [p_{i,j}]$ are Burau matrices then $MP = \left[\sum_k m_{i,k}p_{k,j}\right]$, which is also a Burau matrix, defines the algebra on n generators, $g_1, \dots, g_n$, with

$$g_1^2 = \Big(\sum_k m_{1,k}p_{k,1}\Big)g_1 + \Big(\sum_k m_{1,k}p_{k,2}\Big)g_2 + ... + \Big(\sum_k m_{1,k}p_{k,n}\Big)g_n$$
$$g_2^2 = \Big(\sum_k m_{2,k}p_{k,1}\Big)g_1 + \Big(\sum_k m_{2,k}p_{k,2}\Big)g_2 + ... + \Big(\sum_k m_{2,k}p_{k,n}\Big)g_n$$
$$\vdots$$
$$g_n^2 = \Big(\sum_k m_{n,k}p_{k,1}\Big)g_1 + \Big(\sum_k m_{n,k}p_{k,2}\Big)g_2 + ... + \Big(\sum_k m_{n,k}p_{k,n}\Big)g_n$$

So, if B is the evolution algebra on n generators, $e_1, \dots, e_n$, with

$$e_1^2 = m_{1,1}e_1 + m_{1,2}e_2 + ... + m_{1,n}e_n,$$
$$e_2^2 = m_{2,1}e_1 + m_{2,2}e_2 + ... + m_{2,n}e_n,$$
$$\vdots$$
$$e_n^2 = m_{n,1}e_1 + m_{n,2}e_2 + ... + m_{n,n}e_n$$

and C is defined as the evolution algebra on n generators, $h_1, \dots, h_n$ by

$$h_1^2 = p_{1,1}h_1 + p_{1,2}h_2 + ... + p_{1,n}h_n,$$
$$h_2^2 = p_{2,1}h_1 + p_{2,2}h_2 + ... + p_{2,n}h_n,$$
$$\vdots$$
$$h_n^2 = p_{n,1}h_1 + p_{n,2}h_2 + ... + p_{n,n}h_n$$

then $A = B * C$ is an evolution algerbra. Also note that A is defined by the Burau matrix M and B is defined by the Burau matrix P. This can be seen by the definition of multiplication. Because of this there is a homomorphism $\phi : \mathbf{M_n^b} \to E_{\mathbb{R}[t,t^{-1}]}$ defined by $\phi(M) = A$, where $M = [m_{i,j}]$ and A is the evolution algebra on n generators, $e_1, e_2, \ldots, e_n$, with

$$
\begin{aligned}
e_1^2 &= m_{1,1}e_1 + m_{1,2}e_2 + ... + m_{1,n}e_n, \\
e_2^2 &= m_{2,1}e_1 + m_{2,2}e_2 + ... + m_{2,n}e_n, \\
&\vdots \\
e_n^2 &= m_{n,1}e_1 + m_{n,2}e_2 + ... + m_{n,n}e_n.
\end{aligned}
$$

We have just shown that $\phi(MP) = \phi(M)\phi(P)$.

2.3 The Group of Evolution Algebras

We need to show that the set of all evolution algebras, which are defined from a Burau matrix, with evolution algebra multiplication, is a group. Note that this set is a subset of the set of all evolution algebras with n generators over $\mathbb{R}[t, t^{-1}]$. We will denote the set of all evolution algebras, over $\mathbb{R}[t, t^{-1}]$, defined by Burau matrices, as $E_{\mathbb{R}[t,t^{-1}]}$. Now we will show that $E_{\mathbb{R}[t,t^{-1}]}$ is a group.

First, show that $E_{\mathbb{R}[t,t^{-1}]}$ is closed under multiplication, i.e. let $A, B \in E_{\mathbb{R}[t,t^{-1}]}$ where A is defined by

$$
\begin{aligned}
e_1^2 &= p_{11}e_1 + p_{12}e_2 + ... + p_{1n}e_n, \\
e_2^2 &= p_{21}e_1 + p_{22}e_2 + ... + e_{2n}e_n, \\
&\vdots \\
e_n^2 &= p_{n1}e_1 + p_{n2}e_2 + ... + p_{nn}e_n
\end{aligned}
$$

where $p_{ij} = a_{i,j}$ where $[a_{i,j}]$ is a Burau matrix, and B is defined by

$$\begin{aligned} g_1^2 &= q_{11}g_1 + q_{12}g_2 + ... + q_{1n}g_n \\ g_2^2 &= q_{21}g_1 + q_{22}g_2 + ... + p_{2n}g_n \\ &\vdots \\ g_n^2 &= q_{n1}g_1 + q_{n2}g_2 + ... + q_{nn}g_n \end{aligned}$$

where $q_{i,j} = b_{i,j}$ where $[b_{i,j}]$ is a Burau matrix.

We must show $A * B \in E_{\mathbb{R}[t,t^{-1}]}$. We know that $A * B$ is defined by the generators $h_1, h_2, \dots, h_n$, with

$$\begin{aligned} h_1^2 &= \Big(\sum_k p_{1k}q_{k1}\Big)h_1 + \Big(\sum_k p_{1k}q_{k2}\Big)h_2 + ... + \Big(\sum_k p_{1k}q_{kn}\Big)h_n \\ h_2^2 &= \Big(\sum_k p_{2k}q_{k1}\Big)h_1 + \Big(\sum_k p_{2k}q_{k2}\Big)h_2 + ... + \Big(\sum_k p_{2k}q_{kn}\Big)h_n \\ &\vdots \\ h_n^2 &= \Big(\sum_k p_{nk}q_{k1}\Big)h_1 + \Big(\sum_k p_{nk}q_{k2}\Big)h_2 + ... + \Big(\sum_k p_{nk}q_{kn}\Big)h_n. \end{aligned}$$

But, since we know the Burau matrices form a group $\sum_k p_{ik}q_{kj}$ must be entries in a Burau matrix. So we know $A * B$ is defined from some Burau matrix so $A * B \in E_{\mathbb{R}[t,t^{-1}]}$. Now show that $E_{\mathbb{R}[t,t^{-1}]}$ has an identity element, called E. We must show that $E * A = A = A * E$ for all $A \in E_{\mathbb{R}[t,t^{-1}]}$. Define E by $e_1^2 = e_1$, $e_2^2 = e_2, \dots, e_n^2 = e_n$. Using this definition, we can associate E with the identity matrix, which is the Burau matrix of the trivial braid. Thus $E \in E_{\mathbb{R}[t,t^{-1}]}$. Let $A \in E_{\mathbb{R}[t,t^{-1}]}$ then $E * A$ is defined by

$$\begin{aligned} h_1^2 &= \Big(\sum_k p_{1k}q_{k1}\Big)h_1 + \Big(\sum_k p_{1k}q_{k2}\Big)h_2 + ... + \Big(\sum_k p_{1k}q_{kn}\Big)h_n \\ h_2^2 &= \Big(\sum_k p_{2k}q_{k1}\Big)h_1 + \Big(\sum_k p_{2k}q_{k2}\Big)h_2 + ... + \Big(\sum_k p_{2k}q_{kn}\Big)h_n \\ &\vdots \\ h_n^2 &= \Big(\sum_k p_{nk}q_{k1}\Big)h_1 + \Big(\sum_k p_{nk}q_{k2}\Big)h_2 + ... + \Big(\sum_k p_{nk}q_{kn}\Big)h_n, \end{aligned}$$

where the p_{ij}'s are the coefficients of E and the q_{ij}'s are the coefficients of A. But $p_{ij} = 0$ unless $i = j$ when $p_{ii} = 1$. So $\sum_k p_{ik}q_{kj} = p_{ii}q_{ij} = 1 * q_{ij} = q_{ij}$ but then $h_i^2 = q_{i1}h_1 + q_{i2}h_2 + ... + q_{in}h_n$. So $h_i^2 = g_i^2$. Thus $E * A = A$. Similarly, $A * E$ can be defined as

$$\begin{aligned}
h_1^2 &= \Big(\sum_k q_{1k}p_{k1}\Big)h_1 + \Big(\sum_k q_{1k}p_{k2}\Big)h_2 + ... + \Big(\sum_k q_{1k}p_{kn}\Big)h_n \\
h_2^2 &= \Big(\sum_k q_{2k}p_{k1}\Big)h_1 + \Big(\sum_k q_{2k}p_{k2}\Big)h_2 + ... + \Big(\sum_k q_{2k}p_{kn}\Big)h_n \\
&\vdots \\
h_n^2 &= \Big(\sum_k q_{nk}p_{k1}\Big)h_1 + \Big(\sum_k q_{nk}p_{k2}\Big)h_2 + ... + \Big(\sum_k q_{nk}p_{kn}\Big)h_n
\end{aligned}$$

, where the p_{ij}'s are the coefficients of E and the q_{ij}'s are the coefficients of A. Nevertheless $p_{ij} = 0$ unless $i = j$ when $p_{ii} = 1$. Thus $\sum_k q_{ik}p_{kj} = q_{ij}$ and so $A * E = A$. So E as defined is the identity in $E_{\mathbb{R}[t,t^{-1}]}$.

Next we show that all elements of $E_{\mathbb{R}[t,t^{-1}]}$ have inverses. Assume $A \in E_{\mathbb{R}[t,t^{-1}]}$. Then we know A is defined by a Burau matrix, call it M. (Note $\phi(M) = A$, in this case.) Since $M \in \mathbf{M_n^B}$, M has an inverse M^{-1} Let $M^{-1} = [a_{i,j}]$ then define A^{-1} as

$$\begin{aligned}
e_1^2 &= a_{1,1}e_1 + a_{1,2}e_2 + ... + a_{1,n}e_n, \\
e_2^2 &= a_{2,1}e_1 + a_{2,2}e_2 + ... + a_{2,n}e_n, \\
&\vdots \\
e_n^2 &= a_{n,1}e_1 + a_{n,2}e_2 + ... + a_{n,n}e_n
\end{aligned}$$

which is clearly an element of $E_{\mathbb{R}[t,t^{-1}]}$. Now since $MM^{-1} = I$, we know $\phi(MM^{-1}) = \phi(M)\phi(M^{-1}) = AA^{-1} = E$. So inverses exist.

2.4 The Evolution Algebra Representation

Now that we know that $E_{\mathbb{R}[t,t^{-1}]}$ is a group we can define a representation $\tau : B_n \to E_{\mathbb{R}[t,t^{-1}]}$, by $\tau(\sigma_i) = A_i$, where A_i is the evolution algebra on n generators, $e_1, e_2, \ldots, e_n$ defined by:

$$\begin{aligned} e_i^2 &= (1-t)e_i + (t)e_{i+1} \\ e_{i+1}^2 &= e_i \\ e_j^2 &= e_j \text{ If } j \neq i, i+1. \end{aligned}$$

Also $\tau(\sigma_i^{-1})$ is the evolution algebra on n generators, $e_1, e_2, \ldots, e_n$ defined by:

$$\begin{aligned} e_i^2 &= (t^{-1})e_{i+1} \\ e_{i+1}^2 &= (t^{-1})e_i + (1-t^{-1})e_{i+1} \\ e_j^2 &= e_j \text{ If } j \neq i, i+1. \end{aligned}$$

By our previous theorem, we know that this is a homomorphism on the entire free group on $n-1$ generators. Since the braid group is not free, this definition may not be well defined on the braid group. To verify that it is and therefore will induce a homomorphism on the entire group, we must show that the relations of the braid group are in $ker(\tau)$.

First, we will check that $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} \in ker(\tau)$ for $|i-j| \geq 2$. This means that $\tau(\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}) = \tau(1) = E$ or that $\tau(\sigma_i\sigma_j) = \tau(\sigma_i)\tau(\sigma_j)$. Since we know that τ is a homomorphism on the free group, $\tau(\sigma_i\sigma_j) = \tau(\sigma_i)\tau(\sigma_j)$. We also know $\tau(\sigma_i)$ is defined by

$$\begin{aligned} e_i^2 &= (1-t)e_i + (t)e_{i+1} \\ e_{i+1}^2 &= e_i \\ e_k^2 &= e_k \text{ if } k \neq i, i+1. \end{aligned}$$

and $\tau(\sigma_j)$ is defined by

$$\begin{aligned} g_j^2 &= (1-t)g_j + (t)g_{j+1} \\ g_{j+1}^2 &= g_j \\ g_k^2 &= g_k \text{ if } k \neq j, j+1. \end{aligned}$$

So $\tau(\sigma_i)\tau(\sigma_j)$ is defined as

$$\begin{aligned} h_i^2 &= (1-t)h_i + (t)h_{i+1} \\ h_{i+1}^2 &= h_i \\ h_j^2 &= (1-t)h_j + (t)h_{j+1} \\ h_{j+1}^2 &= h_j \\ h_k^2 &= h_k \text{ if } k \neq i, i+1, j, j+1 \end{aligned}$$

since $|i-j| \geq 2$. If we define $\tau(\sigma_j)$ and $\tau(\sigma_i)$ as we did before then $\tau(\sigma_i)\tau(\sigma_j)$ is defined as

$$\begin{aligned} h_i^2 &= (1-t)h_i + (t)h_{i+1} \\ h_{i+1}^2 &= h_i \\ h_j^2 &= (1-t)h_j + (t)h_{j+1} \\ h_{j+1}^2 &= h_j \\ h_k^2 &= h_k \text{ if } k \neq i, i+1, j, j+1. \end{aligned}$$

for all $|i-j| \geq 2$. Since $\tau(\sigma_i)\tau(\sigma_j) = \tau(\sigma_i)\tau(\sigma_j)$ for all $|i-j| \geq 2$, $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} \in ker(\tau)$ for $|i-j| \geq 2$. Now we know one relation holds.

New we must show that $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} \in ker(\tau)$. As be for we can show this is true by showing $\tau(\sigma_i\sigma_{i+1}\sigma_i) = \tau(\sigma_{i+1}\sigma_i\sigma_{i+1})$ for $i = 1, 2, \cdots, n-2$. We know $\tau(\sigma_i\sigma_{i+1}\sigma_i) = \tau(\sigma_i)\tau(\sigma_{i+1})\tau(\sigma_i)$, since τ is a homomorphism on the free group. Then, if we define $\tau(\sigma_i)$ as before and $\tau(\sigma_{i+1})$ as the evolution algebra on n generators, $g_1, \ldots, g_n$ by

$$\begin{aligned} g_{i+1}^2 &= (1-t)g_{i+1} + (t)g_{i+2} \\ g_{i+2}^2 &= g_{i+1} \\ g_k^2 &= g_k \text{ if } k \neq i+1, i+2. \end{aligned}$$

Then $\tau(\sigma_i)\tau(\sigma_{i+1})$ is defined as the evolution algebra on n generators, $h_1, \ldots, h_n$ by

$$\begin{aligned} h_i^2 &= (1-t)h_i + (t-t^2)h_{i+1} + (t^2)h_{i+2} \\ h_{i+1}^2 &= h_i \\ h_{i+3}^2 &= h_{i+1} \end{aligned}$$

and then $\tau(\sigma_i)\tau(\sigma_{i+1})\tau(\sigma_i)$ is defined as the evolution algebra on n generators, $j_1, \ldots, j_n$ by

$$\begin{aligned} j_i^2 &= (1-t)j_i + (t-t^2)j_{i+1} + (t^2)j_{i+2} \\ j_{i+1}^2 &= (1-t)j_i + (t)j_{i+1} \\ j_{i+3}^2 &= j_i \end{aligned}$$

Now let's look at $\tau(\sigma_{i+1}\sigma_i\sigma_{i+1}) = \tau(\sigma_{i+1})\tau(\sigma_i)\tau(\sigma_{i+1})$. Define $\tau(\sigma_i)$, and $\tau(\sigma_{i+1})$ as before. Now we can define $\tau(\sigma_{i+1})\tau(\sigma_i)$ as the evolution algebra on n generators, $a_1, \ldots, a_n$ by

$$\begin{aligned} a_i^2 &= (1-t)a_i + (t)a_{i+1} \\ a_{i+1}^2 &= (1-t)a_i + (t)a_{i+2} \\ a_{i+3}^2 &= a_i \end{aligned}$$

And so $\tau(\sigma_{i+1})\tau(\sigma_i)\tau(\sigma_{i+1})$ is defined as the evolution algebra on n generators, $b_1, \ldots, b_n$ by

$$\begin{aligned} b_i^2 &= (1-t)b_i + (t-t^2)b_{i+1} + (t^2)b_{i+2} \\ b_{i+1}^2 &= (1-t)b_i + (t)b_{i+1} \\ b_{i+3}^2 &= b_i \end{aligned}$$

Thus $\tau(\sigma_i\sigma_{i+1}\sigma_i) = \tau(\sigma_{i+1}\sigma_i\sigma_{i+1})$ for $i = 1, 2, \cdots, n-2$ and so $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} \in ker(\tau)$. In both cases our relations hold and we know τ is a well defined function which induces a homomorphism on the braid group. In fact, we know that τ is a representation of $\mathbf{B_n}$ by evolution algebras.

2.5 Consequences of the Representation

Now that we know we have a representation we can discuss a few consequences of this representation. Since it is derived so heavily from the Burau representation we know it inherits some of its properties. One of these would be faithfulness, since a braid which is mapped to the zero matrix, is still the product of generators. The same product of generators sent to evolution algebra will still be multiplied in the same way and will produce the zero evolution algebra. Thus our new representation is faithful for $\mathbf{B_3}$, not faithful for $\mathbf{B_n}$, where $n \geq 5$, and unknown for $\mathbf{B_4}$.

2.6 Problems for Future Study

(1) The set of all n dimensional evolution algebras is a category, and the image of the braid group $\mathbf{B_n}$ is a subcategory. How can we characterize this subcategory?

(2) If the images of two braids are homomorphic (in algebra), what can we say about these two braids?

(3) If the images of two braids are isomorphic (in algebra), what can we say about these two braids?

(4) In algebras, we have classification theorem for evolution algebras, can we have some sort of classification about n-braids? This will be new results for braid groups.

(5) More detailed, what is the counterpart concept of algebraic persistence in braid groups? What is the counterpart concept of algebraic transience in braid group? How can we define periodicity for a braid by using its image in algebras?

(6) We are interested in not only the whole braid group, but also classes of braids and even a single braid, evolution algebras provide a tool to make this study possible. Algebras have more algebraic structures than groups do. This is why we "imbed" braid groups into a algebra category. Hecke algebra is one type of algebra generated from braid groups. We hope evolution algebras have more structures than Hecke algebras do when we study braid groups. After we understand these questions, we can move to a good level, probably bring a new landscape for braid groups study.

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