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Holographic Non-perturbative Thermodynamic Systems

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Holographic Non-perturbative Thermodynamic Systems

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by

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Abstract

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence conjectures a duality between field theories and higher dimensional theories of gravitation. Recent results describing thermodynamic systems in the AdS/CFT context include an exact description of the efficiency of black hole heat engines, suggesting questions regarding the nature of heat engines within this formulation and the extent to which thermodynamic principles may be applied. We verify the Clausius statement and the maximum efficiency of the Carnot engine, and show that these follow from the thermodynamic definitions of the heat engine.

In a related scope, we propose that, given the Ryu-Takayanagi prescription for holographic entanglement entropy, the null-energy condition for Poincaré invariant spaces implies the monotonicity of entanglement entropy with respect to the energy scale of the corresponding gauge theory. We examine the implications of this conjecture and describe our work toward its verification.

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Contents

Chapter 1

Introduction

By proposing a duality between quantum field theories and higher-dimensional theories of gravitation, the Anti-de Sitter/Conformal Field Theory (AdS/CFT) Correspondence, or gauge-gravity duality, allows a non-perturbative (*i.e.* non-approximate) exploration of field theories through examining corresponding properties in the higher-dimensional picture. In this project, we are interested in properties of quantum fields that are monotonic with respect to some parameterization or transformation of the theory in question.

Principally, we consider two types of "entropy" in the AdS/CFT context. In Chapter 2, we consider a "holographic heat engine" constructed via black hole thermodynamics, and determine the existence of the Second Law of Thermodynamics in this context. In Chapter 3, we consider entanglement entropy, as described by the Ryu-Takayanagi Conjecture, and propose entanglement entropy as a monotonic function of energy scale, given some parameterized chain of subsystems in the quantum field and conditions on the geometry of the higher-dimensional dual theory. Here, our efforts outline a program toward a rigorous result and demonstrate some intermediary claims, as well as examine the implications of our conjecture.

The remainder of the present chapter briefly introduces the concepts we will discuss in this work. Sections 1.1 and 1.2 review AdS/CFT Correspondence and black hole thermodynamics respectively; in Section 1.3, we consider the Ryu-Takayanagi Conjecture, and finally in Section 1.4 we introduce an example of a monotonic function (with respect to energy scale) in the AdS/CFT picture.

1.1 AdS/CFT correspondence

In 1998 Juan Maldacena conjectured the following [1]:

Conjecture 1.1.1. *In the limit of large N, type IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills on $\mathbb{R}^{1,3}$.*

Type IIB string theory is a theory of supergravity; a supersymmetric Yang-Mills theory is a (conformal) gauge theory. Along with papers by Witten [2] and (independently) Gubser, Klebanov, and Polyakov [3], which described explicitly the dictionary connecting the dual theories, Maldacena's paper heralded a new body of work exploring a *gauge-gravity* correspondence between gauge theories in flat space and higher-dimensional theories of gravitation.

The gauge-gravity correspondence is one of the principle tools that has emerged in the search for non-perturbative descriptions of theories of quantum gravity. The conjectured duality provides powerful insight into otherwise intractable problems: if a theory without gravity is well understood, the correspondence allows the dual theory with gravity to be understood in kind, or *vice versa*. Importantly, in the correspondence the dual theories are "strong-weak", *i.e.* the correspondence allows a strongly-coupled theory to be probed via its weakly-coupled dual.

1.1.1 D-branes and p-branes

Maldacena's AdS/CFT correspondence arises from a relationship between two very different objects in Superstring Theory [see, *e.g.* 4], Dp-branes and extremal p-branes. Superstring Theory concerns the dynamics of *massless, relativistic strings*. Its spectrum includes a massless spin-2 particle (*i.e.* a graviton), hence it is a theory of quantum gravity. Interestingly, Superstring Theory is consistent (in fact, Lorentz-invariant) only in $d = 9 + 1$ dimensions.

We may express a relativistic string as a map $X : [0,\infty) \times [0,\pi] \to \mathbb{R}^{n-1}$ where $X^\mu(\tau,\sigma)$ is parameterized by time-like τ and space-like σ and $\mu = 1,\ldots,d-1$. The classical string action is

$$S_{\text{polyakov}} = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-g} g_{ab} \partial^a X_\mu \partial^b X^\mu \tag{1.1}$$

Strings may be *open* or *closed*. Given the action above, we can show that open strings obey the following boundary conditions (at endpoints $\sigma = 0, \pi$):

$$\text{Neumann:} \qquad \partial_\mu X_\nu = 0 \tag{1.2}$$

$$\text{Dirichlet:} \qquad \partial_t X_\nu = 0 \tag{1.3}$$

where Dirichlet boundary conditions "fix" the endpoint X_ν in space. Consider a string whose endpoint boundary conditions are mixed, *i.e.* for $X^0, X^1, \ldots, X^p$ Neumann boundary conditions hold, and for $X^{p+1}, X^{p+2}, \ldots, X^n$ the Dirichlet boundary conditions hold. Then the endpoints are confined to a p-dimensional surface; this surface defines a *Dp-brane*. We note that these objects are not unfamiliar: D0-branes are points, D1-branes are strings, etc. We will be interested in *D3-branes*, which are 3-dimensional hypersurfaces.

In contrast, p-branes are solutions to supergravity, with the metric

$$ds^2 = H_p^{-\frac{1}{2}}(r) \left(-f(r)t^2 + \sum_{i=1}^{p} (dx^i)^2 \right) + H_p^{\frac{1}{2}}(r)(f^{-1}(r)dr^2 + r^2 d\Omega_{n-p-2}^2) \tag{1.4}$$

where $f(r) = 1 - \frac{r_0}{r}$ and $H(r) = 1 + \left(\frac{r_p}{r}\right)^{n-p-3}$. We note that there is a horizon at $r = r_0$; we will call a p-brane *extremal* as $r_0 \to 0$. Polchinski [5] showed that Dp-branes are equivalent to extremal p-branes; it is precisely through this connection that Maldacena's conjecture is motivated.

Surprisingly, one can show that D-branes give rise to gauge fields. Consider a collection of N incident D3-branes. The configurations of strings whose endpoints terminate on this collection gives rise to N^2 massless particle states. If we consider the low energy limit (*i.e.* as the string length scale approaches zero), this configuration produces a $U(N)$ super-Yang-Mills theory. In particular, in 4 dimensions with a $\mathcal{N}=4$ supersymmetry, the theory is $\mathcal{N}=4$ $SU(N)$ super-Yang-Mills.

This theory consists of both closed strings in the bulk (*i.e.* free from D3-brane boundaries) and open strings affixed to the D3-branes, and their interactions. In the low energy limit, the interaction terms disappear, *i.e.* the boundary and bulk theories decouple. Moreover, we have (i) from the above we find $\mathcal{N}=4$ $SU(N)$ super-Yang-Mills on the boundary, and (ii) free supersymmetric gravity in the bulk. Near the boundary $r_0 \to 0$, the bulk metric becomes that of $\mathrm{AdS}_5 \times S^5$.

1.2 Black hole thermodynamics

Given a static AdS black hole, Hawking and Bekenstein related black hole entropy S and temperature T to horizon area A and surface gravity κ respectively, where

$$S = \frac{A}{4}, \qquad T = \frac{\kappa}{2\pi}. \tag{1.5}$$

Relating total energy with mass, we develop thermodynamics of an (A)dS black hole [6], with a first law

$$\delta U := \delta M = T\delta S + \Omega\delta J + \Phi\delta Q \tag{1.6}$$

where Ω and Φ are the angular velocity and electrostatic potential, respectively.

Kastor, Ray, and Traschen [7] extend black hole thermodynamics to include a pressure as a function of Λ, the cosmological constant:

$$p = -\frac{\Lambda}{8\pi} \tag{1.7}$$

We again have a first law, where M is instead associated with enthalpy H:

$$\delta H := \delta M = T\delta S + V\delta p + \Omega\delta J + \Phi\delta Q \tag{1.8}$$

Summarizing the above results, we may relate the mass M, surface gravity κ, and area A of a black hole to its energy U, temperature T, and entropy in the following way [8]:

- $T = \frac{\kappa}{2\pi}$
- $S = \frac{A}{4} = \pi r_h^2$
- $p = -\frac{\Lambda}{8\pi}$
- $M = H$

Upon these definitions, Johnson [8] develops his notion of a "black hole heat engine". We will explore thermodynamic entropy and the Second Law in this context in Chapter 2.

1.3 The Ryu-Takayanagi Conjecture

In 2006 Shinsei Ryu and Tadashi Takayanagi [9] proposed an equality (up to constants) between the entanglement entropy of a subsystem A of a conformal field theory and the area of a *minimal surface* extending from ∂A in the (static) AdS dual space. That is,

Conjecture 1.3.1. *Let A be a subsystem of a CFT on $\mathbb{R}^{1,d}$, with spatial boundary $\partial A \subset \mathbb{R}^d$; let the CFT reside on the boundary of a (static) dual space AdS_{d+2}. Then the entanglement entropy $S(A)$ is as follows:*

$$S(A) = \frac{1}{4G_N} \min_{m \sim A} \text{area}(m) \tag{1.9}$$

where G_N is the $d = n+2$ dimensional Newton constant and $m \sim A$ if m is homologous to A in AdS_{d+2}.

Ryu and Takayangi demonstrated the above for $d = 1$, *i.e.* CFT_2 embedded in AdS_3, and postulated the conjecture for higher-dimensional static spacetimes. We briefly review entanglement entropy and discuss the condition of homology expressed in the conjecture.

1.3.1 Entanglement entropy

Consider some factorable Hilbert space $\mathcal{H} = \bigotimes_i \mathcal{H}_i$, with density matrix ρ. We may define *reduced density matrices* ρ_i for each subspace $\mathcal{H}_i$ as follows:

$$\rho_i = \mathrm{Tr}_{\mathcal{H}_i'}(\rho) \tag{1.10}$$

where $\mathcal{H}_i' = \bigotimes_{j \neq i} \mathcal{H}_j$. Then we define the *entanglement entropy* for each subsystem $\mathcal{H}_i$ as the *von Neumann* entropy of its reduced density matrix ρ_i, *i.e.*:

$$S(\rho_i) = -\mathrm{Tr}(\rho_i \ln \rho_i) \tag{1.11}$$

A natural choice for independent subsystems is to define subsystems on disjoint regions of space $A, B, \ldots$; this choice is particularly relevant because of the geometric nature of the Ryu-Takayanagi Conjecture. For convenience, we will denote $S(A) = S(\rho_A)$. Generally, entanglement entropy has the property of *strong subadditivity*, which may be described as follows. Let A, B be spatial subsystems. Then the strong subadditivity of entanglement entropy implies the following relations:

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B) \tag{1.12}$$

$$S(A) + S(B) \geq S(A \setminus B) + S(B \setminus A) \tag{1.13}$$

1.3.2 Homology and minimal surfaces

The form of the Ryu-Takayanagi Conjecture expressed in Conjecture 1.3.1 is due to [10]. We recall that in the AdS/CFT picture, we associate each (spatial) subsystem with its dual region on the boundary of the bulk theory. In the language of [10], the minimal surface corresponding to this dual region is the minimum-area surface of the same homology class. Determining the minimal surface is therefore a question of finding an area-minimizing "integral current" for the relevant homology class.

Given that the bulk space is a Riemannian manifold (that is, with positive a definite metric), then such an integral current is known to exist for each homology class of the manifold. However, for reasons of simplicity, we prefer to work with homotopically equivalent surfaces, rather than homologous surfaces. Hence, in Chapter 3, we will consider minimal surfaces that are homotopically equivalent to, and share the boundary of, the subsystem region. We note that in [11], it is shown that for surfaces of dimension $m \geq 3$, the problem of determining minimal surfaces under this homotopic definition is equivalent to the homology problem.

If Conjecture 1.3.1 accurately describes entanglement entropy, then we require that it exhibits strong subadditivity. Headrick and Takayanagi [12] have shown that Conjecture 1.3.1 demonstrates strong subadditivity for static spacetimes. This result follows solely from the homology and minimality conditions, hence does not rely upon the geometry or topology of the bulk spacetime.

The Ryu-Takayanagi conjecture may be generalized to *non-static* (*i.e.* time-dependent) backgrounds by substituting *minimal* surfaces for *extremal* surfaces [10]; in this case, it is conjectured that strong subadditivity requires the *null-energy condition*,

$$K^\mu T_{\mu\nu} K^\nu \geq 0 \tag{1.14}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the bulk spacetime and K^μ is a light-like ("null") vector; this condition suggests a result concerning the monotonicity of trace anomaly coefficients, discussed in Section 1.4.

In Chapter 3, we consider a type of "monotonicity over inclusion" for entanglement entropy, which we conjecture is implied by the Ryu-Takayanagi Conjecture and the null-energy condition.

1.4 Trace anomaly coefficients and a $\mathcal{C}$-theorem

In conformal field theories, we note that classically, the stress energy tensor $T_{\alpha\beta}$ has vanishing trace, which follows readily from its definition nature of the conformal symmetry of the theory. In quantum theories, however, it is often difficult to preserve the tracelessness of $T_{\alpha\beta}$, even in the limits where conformal invariance is recovered, *i.e.* often there exists a non-zero *trace anomaly*

$$\Theta = \langle T^{\mu}_{\mu} \rangle \neq 0 \tag{1.15}$$

The trace anomaly of a theory is of particular interest, and its general form is characterized by several coefficients. We will concern ourselves with a, the coefficient of the Euler term in the trace anomaly, and the *central function* c.

Given a general $n+2$-dimensional Poincaré invariant space, with metric

$$ds^2 = e^{2A(r)} dX^{\rho} dX^{\sigma} \eta_{\rho\sigma} + (dX^r)^2 \tag{1.16}$$

Freedman, Gubser, Pilch, and Warner [13] write a function

$$\mathcal{C}(X^r) = \frac{\mathcal{C}_0}{(A'(X^r))^n} \tag{1.17}$$

which coincides with c and a for appropriate definitions of those coefficients and $\mathcal{C}_0$. We note that X^r can be regarded as a measure of energy scale for corresponding gauge fields. It is shown the null-energy condition is exactly equivalent to the monotonicity of $A'(X^r)$ with respect to X^r, and hence the monotonicity of $\mathcal{C}$ with respect to X^r, or equivalently, renormalization group flow.

In Chapter 3, we conjecture that monotonicity of $A'(X^r)$ (or equivalently, the null-energy condition) implies a similarly monotonic function, $\mathcal{S}(X^r)$, related to the entanglement entropy of a chain of subsystems of a given gauge theory.

Chapter 2

Holographic heat engines and the Second Law

From the prescription in [8] we may construct a black hole "heat engine" (a closed cycle in the p-V plane) with the following thermodynamic variables:

- $T = \frac{\kappa}{2\pi}$, where κ is the surface gravity.
- $S = \frac{A}{4} = \pi r_h^2$.
- $p = -\frac{\Lambda}{8\pi}$.
- $V = \frac{4}{3}\pi r_h^3$, *i.e.* the geometric volume of the black hole.
- $M = U + pV$, *i.e.* the "enthalpy" of the black hole.

2.1 Holographic heat engines: p-V cycles

We may define a Carnot cycle from two paths of constant entropy (adiabats) and two paths of constant temperature (isotherms), with efficiency

$$\eta_{\text{carnot}} = \frac{W}{Q_h} = 1 - \frac{T_c}{T_h} \tag{2.1}$$

In classical thermodynamics, the maximum of the Carnot efficiency is equivalent to the Second Law, or equivalently, the **Clausius statement**:

$$\oint_C \frac{dQ}{T} = \oint_C dS \geq 0 \tag{2.2}$$

where the equality holds for reversible cycles. We are principally concerned with the manifestation of the Second Law in this "holographic" prescription of thermodynamics, hence we seek to show the equivalent statements:

Claim 2.1.1. *For holographic heat engines, (a) the Clausius statement holds, and (b) the Carnot efficiency* η_{carnot} *is maximum.*

We note that $S = S(V) = \pi^{\frac{1}{3}}\left(\frac{3}{4}V\right)^{\frac{2}{3}}$, hence the Clausius statement follows trivially:

$$\oint_C dS = 0 = \oint_C \frac{dQ}{T} \tag{2.3}$$

for any closed path C in the p-V plane.

To address the efficiency of an arbitrary cycle, we will consider p-V cycles in the T-S plane and use a simple "geometric" argument.

2.1.1 Heat engines as T-S cycles

We first must show that **every p-V cycle is closed in T-S**. If T and S are functions of p and V, then the statement is trivial. We have $S = S(V)$. From [7] we have:

$$dM = TdS + Vdp + \Omega dJ + \Phi dq \tag{2.4}$$

Then we may write $M = M(S(V), p, J, q)$. We have $T = \partial_S M(S(V), p, J, q)$, hence

$$T = T(S(V), p, J, q) = T(V, p, J, q). \tag{2.5}$$

Hence, for fixed J, q, if a cycle is closed in p-V, then it is closed in T-S.

From (2.4) we have that dM is exact, hence $\oint_C M = 0$ for any closed C. Then for fixed charge and angular momentum,

$$\oint_C TdS = -\oint_C pdV = -W. \tag{2.6}$$

I.e., for a simple cycle, the magnitude of the work done over a cycle is the area enclosed by C in the T-S plane.

For some simple cycle C,

$$\eta = \frac{W}{Q_h} = 1 - \frac{Q_c}{Q_h}. \tag{2.7}$$

Consider C in the T-S plane and define T_c, T_h as the min/max temperaturates and S_0, S_1 as the min/max entropies of C. Then Q_c, Q_h are the **areas beneath the lower and upper paths of the cycle**, respectively.

At best, we minimize Q_c and maximize Q_h: we reconstruct the Carnot cycle, comprised of two isotherms and two adiabats:

$$\eta \leq 1 - \frac{T_c(S_1 - S_0)}{T_h(S_1 - S_0)} = 1 - \frac{T_c}{T_h} = \eta_{\text{carnot}}. \tag{2.8}$$

Claim 2.1.1 holds.

As a final note, we observe that the Clausius statement and the argument for the maximum efficiency result from the following:

(a) *There exists a First Law.*

(b) *Entropy and temperature are equations of state, i.e. functions of pressure and volume.*

We have that (b) effectively follows from (a) for static black holes; however, we make the statement explicit below.

We are already given that entropy is a function of horizon area, hence geometric volume: $S = S(V)$. Given an AdS black hole with metric

$$ds^2 = -Y(r,\Lambda)dt^2 + \frac{dr^2}{Y(r,\Lambda)} + r^2 d\Omega^2, \tag{2.9}$$

where $Y(r, \Lambda)$ has a maximum root at r_+. Then smoothness at the horizon $r = r_+$ requires

$$\frac{1}{T} = 4\pi Y'(r_+, \Lambda) \tag{2.10}$$

hence $T = T(r_+, \Lambda)$. But $\Lambda = -8\pi p$ and $r_+ = (3\pi V/4)^{1/3}$:

$$T = T(r_+(V), \Lambda(p)) = T(V, p) \tag{2.11}$$

Hence temperature is an equation of state of pressure and volume, as desired.

Chapter 3

Ryu-Takayanagi Conjecture and a $\mathcal{S}$-theorem

We consider a compact $d = n + 2$ Poincaré invariant space, with metric

$$ds^2 = e^{2A(X^r)} dX^\mu dX^\nu \eta_{\mu\nu} + (dX^r)^2$$

where $\eta_{\mu\nu}$ is the d-dimensional Minkowski metric with signature $(-, +, +, \ldots)$, X^r is the (spacelike) radial direction, and X^μ ($\mu = \{0, \ldots, n\}$) are n transverse spacelike dimensions, which comprise a $n+1$-dimensional Minkowski space at constant X^r.

We are given that extension along the dimension X^r corresponds to a change in energy scale for fields embedded in the transverse dimensions. Generally speaking, we are interested in properties of the lower dimensional quantum fields (embedded in $d = n + 1$ slices of constant X^r) that are *monotonic* over renormalization group flow, *i.e.* monotonic with respect to energy scale. We consider entanglement entropy as such a monotonic property, in the following manner. First we fix the timelike coordinate $X^0 = t_0$. Consider some codimension-1 (extended) surface E transverse to X^r (this is a "brane" of constant X^r). Let M be some compact, connected, oriented n-dimensional manifold and let $f_0 : M \to E, f_1 : M \to E$ be two embeddings of M in E such that there exists a homotopy $F : M \times [0, 1] \to E$ between f_0, f_1 with $F[M \times \{\phi\}] \subseteq F[M \times \{\phi'\}]$ whenever $\phi \leq \phi'$. For $\phi \in [0, 1]$, let $f_\phi : M \to E$ denote

$F|_\phi$. Suppose for each f_ϕ, there exists a unique homotopic map $g_\phi \sim f_\phi$ such that g is fixed to the boundary values of f at ∂M and the image of g_ϕ has a minimum n-area for all such homotopic maps (the image of this mapping will constitute our "minimal surface"; see the discussion in Section 1.3.2). Denote $U_\phi := \mathrm{Im}(f_\phi), V_\phi := \mathrm{Im}(g_\phi)$ and let r_ϕ denote the least upper bound of $V_\phi = \mathrm{Im}(g_\phi)$ projected along X^r. By the Ryu-Takayanagi Conjecture, we have

$$S(U_\phi) = \frac{1}{4G_N}\,\mathrm{area}(V_\phi) \tag{3.1}$$

If locally minimal surfaces are globally minimal, then the above suggests that the structure of the bulk geometry influences the entanglement entropy $S(U_\phi)$ only locally to V_ϕ, *i.e.*, r_ϕ sets a relevant scale for $S(U_\phi)$. Hence, given the parameterized chain of regions $\{U_\phi\}_{\phi\in[0,1]}$ on E, we seek (1) a continuous function $\mathcal{S}$ which maps $r_\phi \mapsto S(U_\phi)$, and (2) conditions for the monotonicity of the above function.

To demonstrate the existence and monotonicity of $\mathcal{S}$, we attempt to show the existence of continuous injective functions from $[0,1] \to \mathbb{R}$ mapping $\phi \mapsto S(U_\phi)$ and $\phi \mapsto r_\phi$. Using general topological arguments, in Section 3.1 we suggest that $S(\phi) := S(U_\phi)$ is continuous; we then conjecture that given certain constraints on the bulk metric, S is injective and that r_ϕ is continuous with respect to ϕ, and explore the implications therein. Finally, globally minimal surfaces must also be locally minimal: in Section 3.2 we construct a special case of locally minimal surfaces extending from spherical regions on the brane; eventually, we wish to show that in this special case, our conjectures in Section 3.1 hold.

3.1 The function $\mathcal{S}$

We will first restate the notation above, and state a significant assumption. Again, we consider a compact $d = n + 2$ Poincaré invariant space, with metric $ds^2 = e^{2A(X^r)}dX^\mu dX^\nu \eta_{\mu\nu} + (dX^r)^2$. Let R be a constant time slice (*e.g.* with $X^0 = t_0$) of this space, and note that R is a $(n+1)$-dimensional compact Riemannian manifold, since $ds^2 = e^{2A(X^r)} dX^\mu dX^\nu \delta_{\mu\nu} + (dX^r)^2 \geq 0$, *i.e.* the metric is positive definite over $\mathcal{R}$. Let E be a constant X^r slice of $\mathcal{R}$, and note that E is isometric to $\mathbb{R}^n$.

Let M be some compact, connected, oriented n-dimensional manifold, and let $f_0 : M \to E, f_1 : M \to E$ be two embeddings of M in E such that there exists a homotopy $F : M \times [0,1] \to E$ such that $F(\mathbf{x}, 0) = f_0(\mathbf{x}), F(\mathbf{x}, 1) = f_1(\mathbf{x})$ for all $\mathbf{x} \in M$. Let $f_\phi : M \to E$ such that $f_\phi(\mathbf{x}) = F|_\phi(\mathbf{x}) = F(\mathbf{x}, \phi)$ for $\mathbf{x} \in M$ and $\phi \in [0,1]$. We will assume that the desired homotopic map exists for each f_ϕ (whose image is the "minimal surface" of $\mathrm{Im}(f_\phi)$), *i.e.*

Claim 3.1.1. *For each f_ϕ, $\phi \in [0,1]$, there exists a homotopic map $g_\phi \sim f_\phi$ such that $g_\phi|_{\partial M} = f_\phi|_{\partial M}$ and the image of g_ϕ has a minimum n-area for all such homotopic maps.*

We shall call embeddings with this restricted type of homotopy **fixed-boundary homotopic**, *i.e.*

Definition 3.1.2. *Let f, g be homotopic embeddings of a manifold M with a (possibly empty) boundary ∂M. If $f|_{\partial M} = g|_{\partial M}$, then f, g are **fixed-boundary homotopic**.*

We shall denote $U_\phi := \mathrm{Im}(f_\phi)$ and $V_\phi := \mathrm{Im}(g_\phi)$ for each $\phi \in [0,1]$. We will eventually consider only homotopies F such that $U_\phi \subseteq U'_\phi$ whenever $\phi \leq \phi'$; however, for our discussion of the continuity of entanglement entropy, this assumption is unnecessary.

3.1.1 Continuity of entanglement entropy over deformations

Given the parameterized family of regions $\{U_\phi\}_{\phi \in [0,1]}$ on E, we may express the entanglement entropy of these regions as a real-valued function of the parameter, *i.e.* $S : [0,1] \to \mathbb{R}$ such that $S : \phi \mapsto S(U_\phi)$. We wish to show that S is continuous over $[0,1]$, which by the generality of our choice for f_0, f_1 holds for any homotopic embeddings of sufficiently "nice" manifolds in the appropriate codimension-1 subspaces. We also note that, since entanglement entropy is proportional to the area of the minimal surfaces V_ϕ, it suffices to show that area(V_ϕ) is continuous with respect to ϕ. We wish to prove the following:

Lemma 3.1.3. *Let M be a compact, connected, oriented n-dimensional manifold and let R be a compact $n+1$-dimensional Riemannian manifold with a n-dimensional submanifold E isometric to $\mathbb{R}^n$. If f_0, f_1 are embeddings of M in E homotopic under $F : M \times [0,1] \to E$, then denote $f_\phi := F|_\phi$. If for each f_ϕ there exists fixed-boundary homotopic embedding of M in $\mathcal{R}$, $g_\phi \sim f_\phi$, such that the mapping area* $\text{area}(\text{Im}(g_\phi))$ *is minimum for all fixed-boundary homotopic maps, then define the function $\mathcal{A} : [0,1] \to \mathbb{R}$ such that*

$$\mathcal{A} : \phi \mapsto \text{area}(\text{Im}(g_\phi)) \tag{3.2}$$

Then $\mathcal{A}$ is continuous.

The proof of Lemma 3.1.3 may follow from a simple argument bounding the areas of nearby minimal surfaces V_ϕ. However, we first must demonstrate the continuity of the areas of the regions U_ϕ on E. In particular, we must show the following:

Claim 3.1.4. *Let M be a compact, connected, oriented n-dimensional manifold. If f_0, f_1 are embeddings of M in $\mathbb{R}^n$ homotopic under $F : M \times [0,1] \to \mathbb{R}^n$, then denote $f_\phi := F|_\phi$. Define the function $\sigma : [0,1] \to \mathbb{R}$ such that*

$$\sigma : \phi \mapsto \text{area}(\text{Im}(f_\phi)) \tag{3.3}$$

where $\text{area}(U)$ *denotes the Euclidean n-area of the submanifold $U \subseteq \mathbb{R}^n$. Then σ is continuous, and moreover, for any $\phi \in [0,1]$ and $\epsilon > 0$, there exists some neighborhood of ϕ, $N \subseteq [0,1]$, such that for any $a, b \in N$,* $\text{area}(\text{Im}(f_a) \ominus \text{Im}(f_b)) < \epsilon$.

The above seems reasonable: we would expect that the continuity of the areas $\text{area}(U_\phi)$ would follow from the continuity of F. In particular, for some $\phi \in [0,1]$, suppose $\text{area}(U_\phi) = h_\phi$. Then for any neighborhood N of h_ϕ, we expect there is some neighborhood $D \in [0,1]$ of ϕ such that $\sigma[D] \subseteq N$. Since U_ϕ varies continuously with respect to ϕ, we can imaging U_ϕ (and its area) "changing" as little as we wish, for a small enough neighborhood of ϕ on $[0,1]$. A rigorous proof might benefit from consideration of Lebesgue measures over homotopies, or perhaps more generally, geometric measure theory.

To justify Lemma 3.1.3, first assume that Claim 3.1.4 is true. Then consider any U_a, U_b with corresponding minimal surfaces V_a, V_b. First, we note that the set $U_a \ominus U_b$, where $\ominus$ indicates symmetric difference, has boundary $\partial U_a \cup \partial U_b$. Hence, $W_a = V_b \cup (U_a \ominus U_b)$ is a surface sharing the boundary ∂U_a, and likewise, $W_b = V_a \cup (U_a \ominus U_{\phi'})$ is a surface sharing the boundary ∂U_b. Area is subadditive, hence we have:

$$\begin{aligned} \operatorname{area}(W_a) &\leq \operatorname{area}(V_b) + \operatorname{area}(U_a \ominus U_b) \\ \operatorname{area}(W_b) &\leq \operatorname{area}(V_a) + \operatorname{area}(U_a \ominus U_b) \end{aligned} \tag{3.4}$$

We *would like* to assume that $W_{a,b}$ is fixed-boundary homotopic with $U_{a,b}$; such a claim seems reasonable, since, *e.g.*, W_a shares a boundary with U_a and $V_b \sim U_b \sim U_a$, hence (except for the region $U_a \ominus U_b$) W_a is *almost* homotopic with U_a. **We shall assume the above is true.** But *if* $W_a \sim U_a$ and $W_b \sim U_b$, then the area of, *e.g.*, W_a must be greater than that of the corresponding minimal surface V_a. Hence

$$\begin{aligned} \operatorname{area}(V_a) \leq \operatorname{area}(W_a) &\leq \operatorname{area}(V_b) + \operatorname{area}(U_a \ominus U_{\phi'}) \\ \operatorname{area}(V_b) \leq \operatorname{area}(W_b) &\leq \operatorname{area}(V_a) + \operatorname{area}(U_a \ominus U_b) \end{aligned} \tag{3.5}$$

Combining (3.5), we have the following bound:

$$|\operatorname{area}(V_a) - \operatorname{area}(V_b)| \leq \operatorname{area}(U_a \ominus U_b) \tag{3.6}$$

If given Claim 3.1.4, then we have that $\operatorname{area}(U_a \ominus U_b)$ may be made arbitrarily small over some neighborhood of ϕ. Hence, $|\mathcal{A}(a) - \mathcal{A}(b)| = |\operatorname{area}(V_a) - \operatorname{area}(V_b)|$ may be arbitrarily small: $\mathcal{A}$ is continuous, *i.e.* the entanglement entropy is continuous over deformations of the domain. $\square$

We must note that *the above is not proof*, in actuality. We have made the following non-trivial assumptions: (a) that Claim 3.1.4 holds, and (b) that W_a is homotopic to U_a for any $a \in [0, 1]$. These claims remain to be proven before Lemma 3.1.3 is complete.

3.1.2 Conjecturing the monotonicity of $\mathcal{S}$

We shall now restrict ourselves to homotopies F such that $U_\phi \subsetneq U'_\phi$ whenever $\phi < \phi'$, since there is no hope of monotonicity generally without some notion of "monotonicity" on F. Hence, $\{U_\phi\}_{\phi\in[0,1]}$ is now a parameterized *proper chain* (by inclusion) on E. Again, suppose that there exists a minimal surface V_ϕ for each U_ϕ as described previously.

For the following arguments, we will not make reference to the homotopy function itself (or in fact, to the mappings it relates); consequently, it is more convenient to talk about the **homotopic equivalence** of spaces than homotopy generally.

Definition 3.1.5. *Two spaces U, V are* ***homotopically equivalent*** *if there exists continuous maps $f : U \to V$ and $g : V \to U$ such that $f \circ g$ is homotopic to id_V and $g \circ f$ is homotopic to id_U, where $\mathrm{id}_U, \mathrm{id}_V$ are the identity functions over U, V respectively. We denote this equivalence $U \sim V$.*

In this sense, two spaces are homotopically equivalent if they can be "smoothly deformed" into one another. Given the above, we define a **minimal surface** homotopically equivalent to some space.

Definition 3.1.6. *Let R be a $n+1$ dimensional manifold and $U \subset R$ be a n-dimensional submanifold. Then a submanifold $V \subset R$ is a* ***minimal surface incident on*** *U if V is homotopically equivalent to U, $\partial V = \partial U$, and the n-area of V is minimum for all such homotopically equivalent submanifolds of R.*

Finally, we wish to define some notion of containment for orientable codimension-1 surfaces in R. We note that given some orientable region $U \subset E$ and a minimal surface V incident on E, $\partial U = \partial V$, hence $U \cup V$ is without boundary, *i.e.* it is a closed surface. Then it is sensible to refer to the space *enclosed* by $U \cup V$, *i.e.* the space whose boundary is $U \cup V$. More generally, given some n-submanifold $K \subset R$ with $\partial K \subset E$, then we can suppose that given some appropriate $J \subset E$ with $\partial J = \partial K$, $J \cup K$ is without boundary. If a space H is enclosed by $J \cup K$, then (by an abuse of language), we shall say that H is **enclosed by** K. More precisely:

Definition 3.1.7. *Let $K \subset R$ be a n-dimension oriented manifold with boundary $\partial K \subset E$, and some space $H \subset R$. Then H is* ***enclosed by*** *K if there exists some appropriate $J \subset E$ such that $\partial J = \partial K$ and H is enclosed by $J \cup K$. H is* ***strictly enclosed*** *by K if $K \not\subseteq H$.*

We wish to demonstrate that the function $\mathcal{A} : [0, 1] \to \mathbb{R}$, where $\mathcal{A} : \phi \mapsto \text{area}(V_\phi)$, is injective. Since we posit that $\mathcal{A}$ is a continuous real-valued function, the above implies that $\mathcal{A}$ is monotonic. Interestingly, that $\mathcal{A}$ is injective follows from a somewhat more general conjecture, which also provides several other useful corollaries:

Conjecture 3.1.8. *Let R be a compact $n+1$-dimensional Riemannian manifold with metric $ds^2 = e^{2A(X^r)} dX^\mu dX^\nu \delta_{\mu\nu} + (dX^r)^2$; let E be a constant X^r n-dimensional submanifold E, which is isometric to $\mathbb{R}^n$, and U a compact, connected, oriented n-dimensional submanifold of E. Let $K \subset R$ be any n-dimensional oriented submanifold with $\partial K \subset E$. If $A'(X^r)$ is monotonic, then for any minimal surface V incident on U, if V is strictly enclosed by K, then*

$$\text{area}(V) < \text{area}(K).$$

That is, given some constraints on the metric of R, we expect surfaces strictly enclosing a minimal surface V to have greater area than V. We choose that $A'(X^r)$ must be monotonic, since as a surface extends along X^r, we are interested in how "rapidly" areas transverse to X^r are "warped" by the factor $e^{2A(X^r)}$; presumably this relation determines the shape of a minimal surface, as well as its uniqueness, etc.

Before we state the corollaries of Conjecture 3.1.8, we must first generalize one of the non-trivial assumptions in the previous section. If some n-dimension submanifolds V_1, V_2, with boundaries $\partial V_1, \partial V_2 \subset E$, are homotopically equivalent such that $V_1 \sim V_2$, then we would like to state that if subsets of V_1, V_2 are pasted together such that the resulting submanifold V' has boundary on E, then $V' \sim V_1 \sim V_2$.

Claim 3.1.9. *Suppose $V_1, V_2 \subset R$ are n-dimensional manifolds with boundaries $\partial V_1, \partial V_2 \subset E$ such that $V_1 \sim V_2$. Then if $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ such that $W_1 \cup W_2$ is an n-dimensional manifold and $\partial(W_1 \cup W_2) \subset E$, then $W_1 \cup W_2 \sim V_1 \sim V_2$.*

We state that $W_1 \cup W_2$ must be a manifold of the same dimension as V_1, V_2 because we wish to "paste" subsets only along regions of intersection of V_1, V_2. In any case, it seems the claim is assumed implicitly in the arguments of several papers discussing the minimal surfaces of the Ryu-Takayanagi conjecture.

We are now ready to state some corollaries of Conjecture 3.1.8.

Corollary 3.1.10. *Minimal surfaces of regions $U \subset E$ are unique.*

Suppose V_1, V_2 are both minimal surfaces of some n-dimensional submanifold $U \subset E$. Let H be the intersection of the volumes enclosed by V_1 and V_2 respectively, and $K = \partial H \cap (V_1 \cup V_2)$. $K = W_1 \cup W_2$ for some $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$, since H must be bounded by subsets of V_1, V_2 and U; moreover, $\partial K \subset E$. Hence by Claim 3.1.9, we have $K \sim U$, and we note that $\partial K = \partial U$. Suppose $V_1 \neq V_2$. Then K is not identical to both V_1, V_2; suppose $K \neq V_1$. K is enclosed (by definition) by V_1, hence strictly enclosed. By Conjecture 3.1.8, area$(K) <$ area(V_1). But then V_1 is not minimal, a contradiction. $V_1 = V_2$. □

Corollary 3.1.11. *Let V_ϕ be the minimal surface incident upon the region U_ϕ in the parametrized chain $\{U_\phi\}_{\phi \in [0,1]}$. Then $\{V_\phi\}_{\phi \in [0,1]}$ is a proper chain under enclosure, i.e., for any $V_a, V_b \in \{V_\phi\}$, if $a < b$ then V_b strictly encloses V_a.*

Let $U_a, U_b \in \{U_\phi\}$ such that $a < b$, hence $U_a \subsetneq U_b$; let V_a, V_b be the minimal surfaces incident on U_a, U_b respectively. $\partial V_a = \partial U_a \neq \partial V_b = \partial U_b$, hence $V_a \neq V_b$. Let H be the intersection of the volumes enclosed by V_a and V_b respectively, and $K = \partial H \cap (V_a \cup V_b)$. $K = W_a \cup W_b$ for some $W_a \subseteq V_a$ and $W_b \subseteq V_b$, since H must be bounded by subsets of V_a, V_b and U_a, U_b; moreover, $\partial K \subset E$. Hence by Claim 3.1.9, we have $K \sim U_a$. We note that $U_a \subset U_b$, hence $\partial K = \partial U_a$. Suppose V_a is not strictly enclosed by V_b. Then $K \neq V_a$. K is enclosed (by definition) by V_a, hence strictly enclosed. By Conjecture 3.1.8, area$(K) <$ area(V_a). But then V_a is not minimal, a contradiction. $V_a \subsetneq V_b$. □

Corollary 3.1.12. *$\mathcal{A}$ is injective, hence monotonic.*

Suppose $a, b \in [0, 1]$ and $a \neq b$. Then let $a < b$. Then V_a is strictly enclosed in V_b, hence by Conjecture 3.1.8, $\text{area}(V_a) < \text{area}(V_b)$. Hence, $\mathcal{A}(a) = \text{area}(V_a) < \mathcal{A}(b) = \text{area}(V_b)$. $\mathcal{A}$ is injective and monotonic. $\square$

We now define our function mapping the parameter ϕ to the X^r-extent of the minimal surface V_ϕ of U_ϕ. Let $\mathcal{R} : [0, 1] \to \mathbb{R}$ such that

$$\mathcal{R} : \phi \mapsto \sup(\pi_r[V_\phi]) \tag{3.7}$$

where π_r is the projection map from R onto X^r. We can show the following:

Corollary 3.1.13. *$\mathcal{R}$ is monotonic.*

Suppose $a, b \in [0, 1]$ and $a < b$. Then V_a is strictly enclosed in V_b, hence $\pi_r[V_a] \subseteq \pi_r[V_b]$. Hence, $\sup(\pi_r[V_a]) \leq \sup(\pi_r[V_b])$. $\mathcal{R}(a) \leq \mathcal{R}(b)$. $\square$

However, we need that $\mathcal{R}$ be continuous and injective: even if $\mathcal{R}$ is continuous, we need strict monotonicity to imply injection. Hence the following remain to be shown:

Conjecture 3.1.14. *$\mathcal{R} : [0, 1] \to \mathbb{R}$ is continuous.*

Conjecture 3.1.15. *Given $A'(X^r)$ monotonic, $\mathcal{R}$ is injective.*

In Conjecture 3.1.15, we constrain the metric of R by analogy to Conjecture 3.1.8; some other constraint may be more appropriate. If $\mathcal{R}$ is continuous and injective, then we may restrict $\mathcal{R}$ to the bijection $\tilde{\mathcal{R}} : [0, 1] \to [a, b]$, where $\text{Im}(\mathcal{R}) = [a, b]$ for some real a, b by the continuity of $\mathcal{R}$. We note that $\tilde{\mathcal{R}}, \tilde{\mathcal{R}}^{-1}$ are continuous and monotonic, hence we are given the monotonic continuous function $\mathcal{S} = \mathcal{A} \circ \tilde{\mathcal{R}}^{-1}$; $\mathcal{S}$ is our desired function (up to a scalar factor), since for any $r_\phi \in \text{Im}(\mathcal{R})$, $\mathcal{S}(r_\phi) = \mathcal{A}(\tilde{\mathcal{R}}^{-1}(r_\phi)) = \mathcal{A}(\phi) = \text{area}(V_\phi)$. Figure 3-1 shows the mappings established in this section.

$$
\begin{tikzcd}
{[0,1]} \arrow[r, hook, "\mathcal{A}"] \arrow[d, hook, two heads, "\tilde{\mathcal{R}}"'] & \mathbb{R} \\
\operatorname{Im}\mathcal{R} \arrow[ur, hook, dashed, "\mathcal{S}"'] &
\end{tikzcd}
$$

Figure 3-1: Commutative diagram for $\mathcal{A}, \tilde{\mathcal{R}}$, and $\mathcal{S}$.

3.1.3 A note on the monotonicity of $\mathcal{A}$

We note that Corollary 3.1.12, which (given Conjecture 3.1.8) claims the monotonicity of $\mathcal{A}$ with respect to the parameter ϕ, presents significant physical implications. We recall that $\mathcal{A}(\phi)$ is proportional to the entanglement entropy of the subsystems U_ϕ, *i.e.*

$$
S(U_\phi) = \frac{1}{4G_N}\mathcal{A}(\phi) \tag{3.8}
$$

Hence the monotonicity of $\mathcal{A}$ is equivalent to the monotonicity of $S(U_\phi)$ with respect to ϕ. That is,

Corollary 3.1.16. *Let E be a gauge theory with a AdS/CFT dual with metric $ds^2 = e^{2A(X^r)}dX^\mu dX^\nu \delta_{\mu\nu} + (dX^r)^2$. Suppose $A'(X^r)$ is monotonic. Then given any chain (by inclusion) of subsystems $\{U_\phi\}$, if $U_a \subset U_b$, then $S(U_a) \leq S(U_b)$.*

This claim is fairly strong in of itself; however, we may imagine that any (or at least most) two submanifolds that are homotopically equivalent, and for which one contains the other, admit a deformation that is "monotonic by inclusion", *i.e.* that can give rise to a parameterized chain as described above. Then for *any* two spatial subsystems A, B where $A \subseteq B$, if the corresponding regions U_A, U_B on the boundary of the bulk theory are homotopically equivalent, we might expect $S(A) \leq S(B)$. Certainly such a claim may seem intuitive (*i.e.* that a growing spatial subsystem should have non-decreasing entanglement entropy); however, insofar as the author is aware, the statement is unproven.

3.2 Locally minimal surfaces in AdS-like space

We note that globally minimal surfaces must also be *locally minimal*, *i.e.* they must must minimize area for small variations with respect to some parameterization. Hence, we turn our attention to locally minimal surfaces, and examine an ansatz for locally minimal surfaces extending from spherical regions on the "brane" (this is the submanifold E of constant X^r, isometric to Euclidean space), with the eventual goal of demonstrating that the conjectures in Section 3.1 hold in this special case.

Let $g_{\mu\nu}$ be the background metric of a n-dimensional Poincaré invariant space, such that $ds^2 = g_{\mu\nu} dX^\mu dX^\nu = e^{2A(r)} dX^\rho dX^\sigma \eta_{\rho\sigma} + dr^2$ where $\eta_{\rho\sigma}$ is the $(n-1)$-dimensional Minkowski metric with signature $(-,+,+,\ldots)$. Define h_{ab} to be the induced metric of a codimension-1 hypersurface Σ parameterized by $X_\mu = X_\mu(\xi^a)$, where ξ^a are coordinates on the surface. Then

$$h_{ab} d\xi^a d\xi^b = ds^2 = g_{\mu\nu} dX^\mu dX^\nu = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu d\xi^a d\xi^b, \tag{3.9}$$

hence,

$$h_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu = e^{2A(r)} \partial_a X^\rho \partial_b X^\sigma \eta_{\rho\sigma} + \partial_a r \partial_b r. \tag{3.10}$$

We can calculate the area of Σ as follows:

$$\text{area}(\Sigma) = \int d\xi^{n-1} \sqrt{h} \tag{3.11}$$

where $h = \det h_{ab}$. If Σ is locally minimal in area, then (3.11) is extremized for small variations δX^μ, *i.e.* $\delta(\text{area}(\Sigma)) = 0$ with respect to δX^μ. We note that

$$\frac{\partial}{\partial h_{ab}} h = h h^{ab} \tag{3.12}$$

hence $\delta\sqrt{h} = \frac{1}{2}\sqrt{h} h^{ab} \delta h_{ab}$. We calculate:

$$\delta h_{ab} = \delta(g_{\mu\nu}(r) \partial_a X^\mu \partial_b X^\nu) = \partial_\nu g_{\rho\sigma} \partial_a X^\rho \partial_b X^\sigma \delta X^\nu + 2 g_{\mu\nu} \partial_a X^\mu \delta(\partial_b X^\nu) \tag{3.13}$$

Hence,

$$\delta(\text{area}(\Sigma)) = \int d\xi^{n-1}\sqrt{h}h^{ab}\left[\frac{1}{2}\partial_\nu g_{\rho\sigma}\partial_a X^\rho \partial_b X^\sigma \delta X^\nu + g_{\mu\nu}\partial_a X^\mu \delta(\partial_b X^\nu)\right] \tag{3.14}$$

$$= \int d\xi^{n-1}\left[\frac{1}{2}\sqrt{h}h^{ab}\partial_\nu g_{\rho\sigma}\partial_a X^\rho \partial_b X^\sigma - \partial_b(\sqrt{h}h^{ab}g_{\mu\nu}\partial_a X^\mu)\right]\delta X^\nu \tag{3.15}$$

giving the equation of motion

$$\frac{1}{2}\sqrt{h}h^{ab}\partial_\nu g_{\rho\sigma}\partial_a X^\rho \partial_b X^\sigma - \partial_b(\sqrt{h}h^{ab}g_{\mu\nu}\partial_a X^\mu) = 0. \tag{3.16}$$

3.2.1 An ansatz for a minimal surface

At this point, we choose a parameterization for the hypersurface $X^\mu(\xi^a)$. As a first case, we are interested in an $(n-1)$-ball region on the brane; the boundary of the region, an $(n-2)$-sphere, defines the boundary of the minimal surface on the brane. Given $ds^2 = e^{2A(r)}dX^\rho dX^\sigma \eta_{\rho\sigma} + dr^2$, every constant r slice is Minkowski, hence by symmetry of $g_{\mu\nu}$ and of the boundary on the brane, we expect every constant r slice of the minimal surface to be bounded in a $(n-2)$-sphere.

For any n-sphere of radius r embedded in $\mathbb{R}^{n+1}$, the *typical* parameterization follows:

$$\begin{aligned} x^1 &= r\cos\phi_1 \\ x^2 &= r\sin\phi_1\cos\phi_2 \\ &\vdots \\ x^n &= r\sin\phi_1\ldots\sin\phi_{n-1}\cos\phi_n \\ x^{n+1} &= r\sin\phi_1\ldots\sin\phi_{n-1}\sin\phi_n \end{aligned} \tag{3.17}$$

where, by induction, we can easily show that

$$x_i x^i = \sum_{i=1}^{n+1}(x^i)^2 = r^2, \tag{3.18}$$

and that for any $\mathbf{x} \in \mathbb{R}^{n+1}$ such that $|\mathbf{x}| = r^2$, $\mathbf{x}$ is in the image of the parameterization, *i.e.* (3.17) is a valid parameterization of an n-sphere of radius r.

We would like to consider n-spheres generally, hence we introduce a more useful parameterization $\mathbf{x} : \mathbb{R}^{n-1} \to \mathbb{R}^n$ of the $(n-1)$-sphere in $\mathbb{R}^n$, which is roughly analagous to (3.17), but appropriately permutes the parameters to allow a recursive definition.

3.2.2 Another parameterization of the $(n-1)$-sphere

We note that we define a $(n-1)$-sphere as the set of points equidistant (*e.g.* radius r) from the origin in $\mathbb{R}^n$. Let $S_0 \subset \mathbb{R}^1$ be parameterized as $x^1_{(0)} = r\phi^0$, $\phi^0 \in \{-1, 1\}$, and note that $\sum_{k=1}^{1}(x^k_{(0)})^2 = (x^1_{(0)})^2 = r^2(\phi^0)^2 = r^2$, hence all points in S_0 are radius r from the origin. Moreover, S_0 includes all points of radius r from the origin, since $\sqrt{r^2} = \pm r \in \mathbb{R}$. Hence S_0 is a 0-sphere of radius r.

More generally, we may parameterize a $(n-1)$-sphere $S_{n-1} \subset \mathbb{R}^n$ of radius r as follows:

$$\begin{aligned} x^n &= r\cos\phi^{n-1} \\ x^{n-1} &= \sin\phi^{n-1}\tilde{x}^{n-1} \\ x^{n-2} &= \sin\phi^{n-1}\tilde{x}^{n-2} \\ \vdots & \qquad \vdots \\ x^1 &= \sin\phi^{n-1}\tilde{x}^1 \end{aligned} \tag{3.19}$$

where $x^l = x^l_{(n-1)}$ and $\tilde{x}^k = x^k_{(n-2)}(\phi^1, \ldots, \phi^{n-2})$ is the kth coordinate of the $(n-2)$-sphere of radius r. (We note that ϕ^0 only changes the sign of x^1; the same may be achieved by changing the sign of ϕ^1, hence ϕ^0 is redundant for $n \geq 2$.) Then $\sum_{k=1}^{n-1}(\tilde{x}^k)^2 = r^2$, hence

$$\sum_{k=1}^{n}(x^k)^2 = r^2\cos^2\phi^{n-1} + \sin^2\phi^{n-1} = r^2(\cos^2\phi^{n-1} + \sin^2\phi^{n-1}) = r^2. \tag{3.20}$$

Conversely, if there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $|\mathbf{x}| = \sum_{k=1}^{n}(x^k)^2 = r^2$, then $|x^n| \leq r$. Suppose $x^n = \pm r$: then $x^k = 0$, $k \neq n$, hence $x^n = \pm r = r\cos\phi^{n-1}$ and $x^k = \sin\phi^{n-1}|\tilde{x}^k|$ where $\phi^{n-1} = 0, \pi$. Assume $|x^n| < r$. Then we choose ϕ^{n-1} such that $x^n = r\cos\phi^{n-1}$, and choose $\tilde{x}^k$ such that $x^k = \sin\phi^{n-1}\tilde{x}^k$ for $k \neq n$ (we note that $\phi^{n-1} \neq 0, \pi$, hence $\sin\phi^{n-1} \neq 0$). Then $r^2 = \sum_{k=1}^{n}(x^k)^2 = r^2\cos^2\phi^{n-1} + \sin^2\phi^{n-1}\sum_{k=1}^{n-1}(\tilde{x}^k)^2$, hence

$$r^2 = r^2(\cos^2\phi^{n-1} + \sin^2\phi^{n-1}) + \sin^2\phi^{n-1}\left(\sum_{k=1}^{n-1}(\tilde{x}^k)^2 - r^2\right) \tag{3.21}$$

$$r^2 = r^2 + \sin^2\phi^{n-1}\left(\sum_{k=1}^{n-1}(\tilde{x}^k)^2 - r^2\right) \tag{3.22}$$

$$0 = \sin^2\phi^{n-1}\left(\sum_{k=1}^{n-1}(\tilde{x}^k)^2 - r^2\right) \tag{3.23}$$

But $\sin^2\phi^{n-1} \neq 0$, hence $\sum_{k=1}^{n-1}(\tilde{x}^k)^2 = r^2$ and $\tilde{x}^k$ are coordinates of the $(n-2)$-sphere of radius r. Hence (3.19) is a valid parameterization of the $(n-1)$-sphere.

3.2.3 Determining the induced metric h_{ab}

We propose an ansatz for a minimal surface Σ_n in $d = n+2$ Poincaré invariant space, where each slice at constant X^0, X^r is a $(n-1)$-sphere of radius $R = e^{A(\xi^r)}s(\xi^r)$, $\xi^r = X^r$, for some function $s(\xi^r)$. For $d = 1+2$, each slice of contains a 0-sphere, hence define a surface Σ_0 such that:

$$\begin{aligned} X^0 &= t_0 \\ X^r &= \xi^r \\ X^1 &= s(\xi^r)\xi^0 \end{aligned} \tag{3.24}$$

where $\xi^0 \in \{-1, 1\}$ (we note that $X^1X_1 = e^{2A(\xi^r)}s(\xi^r)^2 = R^2$).

More generally, define Σ_n (embedded in $d = n + 2$ dimensions) recursively as follows:

$$\begin{aligned} X^0 &= t_0 \\ X^r &= \xi^r \\ X^n &= s(\xi^r)\cos\xi^{n-1} \\ X^{n-1} &= \sin\xi^{n-1}\tilde{X}^{n-1} \\ \vdots \quad & \quad \vdots \\ X^1 &= \sin\xi^{n-1}\tilde{X}^1 \end{aligned} \tag{3.25}$$

where $\tilde{X}^k = \tilde{X}^k(\xi^1, \ldots, \xi^{n-2}, \xi^r)$ is the kth coordinate of the surface Σ_{n-1} (in $d = (n-1)+2$ dimensions, comprised of $(n-2)$-spheres of radius $R = e^{A(\xi^r)}s(\xi^r)$, and $\tilde{X}^r = X^r = \xi^r$ and $\tilde{X}^0 = X^0 = t_0$). Let $\tilde{h}_{ab}$ be the induced metric of Σ_{n-1} and $\tilde{g}_{\mu\nu}$ the background metric in $d = (n-1)+2$, *i.e.*

$$\tilde{h}_{ab} = \tilde{g}_{\mu\nu}\partial_a\tilde{X}^\mu\partial_b\tilde{X}^\nu = \sum_{\mu=1}^{n-1} e^{2A(\xi^r)}\partial_a\tilde{X}^\mu\partial_b\tilde{X}^\mu + \delta_a^r\delta_b^r \tag{3.26}$$

We seek h_{ab}, the induced metric of Σ_n, in terms of $\tilde{h}_{ab}$. We observe:

$$h_{ab} = g_{\mu\nu}\partial_a X^\mu\partial_b X^\nu \tag{3.27}$$

$$= \sum_{\mu=1}^{n} e^{2A}\partial_a X^\mu\partial_b X^\mu + \delta_a^r\delta_b^r \tag{3.28}$$

$$\begin{aligned} &= \sum_{\mu=1}^{n-1} e^{2A}\partial_a(\sin\xi^{n-1}\tilde{X}^\mu)\partial_b(\sin\xi^{n-1}\tilde{X}^\mu) \\ &\qquad + e^{2A}\partial_a[s(\xi^r)\cos\xi^{n-1}]\partial_b[s(\xi^r)\cos\xi^{n-1}] + \delta_a^r\delta_b^r \end{aligned} \tag{3.29}$$

We note that $a = r, n-1$ or $a < n-1$; similarly for b. We consider all cases. Comparing to (3.26), if $a, b \neq n-1$ and $a, b \neq r$, then

$$h_{ab} = \sin^2(\xi^{n-1})\tilde{h}_{ab} \tag{3.30}$$

Similarly, if $a = r$ and $b \neq r, n-1$, then

$$h_{rb} = \sin^2(\xi^{n-1})\tilde{h}_{rb} + 0 = \sin^2(\xi^{n-1})\tilde{h}_{rb} \tag{3.31}$$

Let instead $a, b = r$. Then we have

$$h_{rr} = \sin^2(\xi^{n-1})(\tilde{h}_{rr} - 1) + 1 + \cos^2(\xi^{n-1})e^{2A}(s'(\xi^r))^2 \tag{3.32}$$

Let $a, b = n-1$. Then:

$$\begin{align}
h_{n-1,n-1} &= \cos^2(\xi^{n-1})e^{2A}\sum_{\mu=1}^{n-1}(\tilde{X}^\mu)^2 + \sin^2(\xi^{n-1})e^{2A}s(\xi^r)^2 \tag{3.33}\\
&= \cos^2(\xi^{n-1})e^{2A}s(\xi^r)^2 + \sin^2(\xi^{n-1})e^{2A}s(\xi^r)^2 \tag{3.34}\\
&= e^{2A}s(\xi^r)^2 \tag{3.35}
\end{align}$$

Let $a = n-1, b \neq n-1$. Then:

$$\begin{align}
h_{n-1,b} &= \cos(\xi^{n-1})\sin(\xi^{n-1})e^{2A}\sum_{\mu=1}^{n-1}\tilde{X}^\mu\partial_b\tilde{X}^\mu \nonumber\\
&\qquad - e^{2A}s(\xi^r)\sin(\xi^{n-1})\partial_b[s(\xi^r)\cos(\xi^{n-1})] \tag{3.36}\\
&= \frac{1}{2}\cos(\xi^{n-1})\sin(\xi^{n-1})e^{2A}\sum_{\mu=1}^{n-1}\partial_b(\tilde{X}^\mu)^2 \nonumber\\
&\qquad - e^{2A}s(\xi^r)\sin(\xi^{n-1})\partial_b[s(\xi^r)\cos(\xi^{n-1})] \tag{3.37}\\
&= \frac{1}{2}\cos(\xi^{n-1})\sin(\xi^{n-1})e^{2A}\partial_b s(\xi^r)^2 \nonumber\\
&\qquad - e^{2A}s(\xi^r)\sin(\xi^{n-1})\partial_b[s(\xi^r)\cos(\xi^{n-1})] \tag{3.38}\\
&= \cos(\xi^{n-1})\sin(\xi^{n-1})e^{2A}s(\xi^r)\partial_b s(\xi^r) \nonumber\\
&\qquad - e^{2A}s(\xi^r)\sin(\xi^{n-1})\partial_b[s(\xi^r)\cos(\xi^{n-1})] \tag{3.39}
\end{align}$$

Hence, if $b = r$, then

$$h_{n-1,r} = \cos(\xi^{n-1})\sin(\xi^{n-1})e^{2A}s(\xi^r)s'(\xi^r) - e^{2A}s(\xi^r)\sin(\xi^{n-1})s'(\xi^r)\cos(\xi^{n-1}) = 0 \tag{3.40}$$

Similarly, if $b < n-1$, then

$$h_{n-1,b} = 0 - 0 = 0 \tag{3.41}$$

We note that h_{ab} is symmetric. From the above, we can write:

$$h_{ab} = \begin{pmatrix} h_{rr} & 0 & \sin^2(\xi^{n-1})\tilde{h}_{rb} & \cdots \\ '' & e^{2A}s(\xi^r)^2 & 0 & \cdots \\ '' & '' & \sin^2(\xi^{n-1})\tilde{h}_{ab} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.42}$$

Determining h_{ab} explicitly

Let $h_{ab}^{(n)}$ be the induced metric of the surface Σ_n, as defined above. By induction on n, we show the following:

Claim 3.2.1. *Let $h_{ab}^{(n)}$ be the induced metric of the surface Σ_n. Then $h_{ab}^{(n)}$ is diagonal, and the following holds:*

(a) $h_{rr} = e^{2A}s'(\xi^r)^2 + 1$

(b) $h_{aa} = e^{2A}s(\xi^r)^2 \prod_{k=1}^{n-1-a} \sin^2(\xi^{n-k})$ *for* $a \neq r$.

First consider $h_{ab}^{(1)}$, the induced metric of Σ_1. We have

$$h_{ab}^{(1)} = h_{rr}^{(1)} = g_{\mu\nu}\partial_r X^\mu \partial_r X^\nu = e^{2A}s'(\xi^r)^2(\xi^0)^2 + 1 = e^{2A}s'(\xi^r)^2 + 1 \tag{3.43}$$

Note that $h_{ab}^{(1)}$ is 1×1, hence diagonal; (a), (b) above are satisfied, the latter trivially since there does not exist an index $a \neq r$. Assume Claim 3.2.1 holds for $h_{ab}^{(n)}$; consider $h_{ab} := h_{ab}^{(n+1)}$, and let $\tilde{h}_{ab} := h_{ab}^{(n)}$. We note that off-diagonal elements h_{cd}, $c \neq d$, are either zero or products of $\tilde{h}_{cd}$, *i.e.* $h_{cd} = 0, \sin^2(\xi^{n-1})\tilde{h}_{cd}$. But if $c \neq d$, then $\tilde{h}_{cd} = 0$

since $\tilde{h}_{ab}$ is diagonal. Hence $h_{cd} = 0$: h_{ab} is diagonal. Moreover,

$$h_{rr} = \sin^2(\xi^{n-1})(\tilde{h}_{rr} - 1) + 1 + \cos^2(\xi^{n-1})e^{2A}(s'(\xi^r))^2 \tag{3.44}$$

$$= \sin^2(\xi^{n-1})(e^{2A}(s'(\xi^r))^2 + 1 - 1) + 1 + \cos^2(\xi^{n-1})e^{2A}(s'(\xi^r))^2 \tag{3.45}$$

$$= \sin^2(\xi^{n-1})e^{2A}(s'(\xi^r))^2 + \cos^2(\xi^{n-1})e^{2A}(s'(\xi^r))^2 + 1 \tag{3.46}$$

$$= e^{2A}(s'(\xi^r))^2 + 1 \tag{3.47}$$

Finally, $h_{nn} = e^{2A}s(\xi^r)^2 = e^{2A}s(\xi^r)^2 \prod_{k=1}^{(n+1)-1-n} \sin^2(\xi^{n-k})$ and for $a < n$,

$$h_{aa} = \sin^2(\xi^n)\tilde{h}_{aa} = \sin^2(\xi^n)e^{2A}s(\xi^r)^2 \prod_{k=1}^{n-1-a} \sin^2(\xi^{n-k}) \tag{3.48}$$

$$= e^{2A}s(\xi^r)^2 \prod_{k=1}^{(n+1)-1-a} \sin^2(\xi^{n+1-k}) \tag{3.49}$$

Claim 3.2.1 is proven by induction. □

From the above, we write:

$$h_{ab} = \begin{pmatrix} e^{2A}s'(\xi^r)^2 + 1 & & & & \\ & e^{2A}s(\xi^r)^2 & & & \\ & & \sin^2(\xi^{n-1})e^{2A}s(\xi^r)^2 & & \\ & & & \sin^2(\xi^{n-1})\sin^2(\xi^{n-2})e^{2A}s(\xi^r)^2 & \\ & & & & \ddots \end{pmatrix} \tag{3.50}$$

with off-diagonal elements 0.

3.2.4 Constraining the ansatz

Given our spherical ansatz Σ_n, parameterized in (3.25), and its induced metric h_{ab} in (3.50), we may attempt to solve our equation of motion for locally minimal surfaces (3.16)

$$\frac{1}{2}\sqrt{h}h^{ab}\partial_\nu g_{\rho\sigma}\partial_a X^\rho \partial_b X^\sigma - \partial_b(\sqrt{h}h^{ab}g_{\mu\nu}\partial_a X^\mu) = 0$$

in terms of the function $s(\xi^r)$. First we note generally the following:

$$\begin{aligned}
\partial_\gamma h_{ab} &= \partial_\gamma (h_{ab}) && (3.51)\\
&= \partial_\gamma \left(\sum_{\rho=1}^{n-1} e^{2A(\xi^r)} \partial_a X^\rho \partial_b X^\rho + \delta_a^r \delta_b^r \right) && (3.52)\\
&= 2A'(\xi^r)\delta_\gamma^r \sum_{\rho=1}^{n-1} e^{2A(\xi^r)} \partial_a X^\rho \partial_b X^\rho && (3.53)\\
&= 2A'(\xi^r)(h_{ab} - \delta_a^r \delta_b^r)\delta_\gamma^r && (3.54)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial(\partial_k X^\gamma)} h_{ab} &= \frac{\partial}{\partial(\partial_k X^\gamma)} \left(g_{\rho\sigma} \partial_a X^\rho \partial_b X^\sigma \right) && (3.55)\\
&= 2 g_{\rho\sigma} \partial_a X^\rho \delta_b^k \delta_\gamma^\sigma && (3.56)
\end{aligned}$$

We consider the first term, and note that $\partial_\nu(\partial_a X^\mu) = 0$, hence:

$$\partial_\nu (g_{\rho\sigma}) \partial_a X^\rho \partial_b X^\sigma = \partial_\nu \left(g_{\rho\sigma} \partial_a X^\rho \partial_b X^\sigma \right) = \partial_\nu h_{ab} = 2A'(\xi^r)(h_{ab} - \delta_a^r \delta_b^r)\delta_\nu^r \qquad (3.57)$$

and we have

$$\frac{1}{2}\sqrt{h} h^{ab} \partial_\nu g_{\rho\sigma} \partial_a X^\rho \partial_b X^\sigma = A'(\xi^r)\sqrt{h} h^{ab}(h_{ab} - \delta_a^r \delta_b^r)\delta_\nu^r = A'(\xi^r)\sqrt{h}(n - h^{rr})\delta_\nu^r \quad (3.58)$$

It remains to calculate the second term of our equation of motion (3.16), and simplify in terms of $s(\xi^r)$; these calculations shall be left as a future project.

As a final note, in the second term, h and h^{ab} may be thought as functions of h_{ab}. Hence, we may expand $\partial_b h$ and $\partial_b h^{ab}$ in terms of $\partial_b h_{ab}$, which we compute below. Note that $h_{ab} = h_{ab}(g_{\mu\nu}(X^\rho), \partial_a X^\sigma) = h_{ab}(X^\rho, \partial_a X^\sigma)$. Hence,

$$\partial_b h_{cd} = \partial_\gamma h_{cd} \partial_b X^\gamma + \frac{\partial}{\partial(\partial_k X^\gamma)} h_{cd} \partial_b \partial_k X^\gamma \qquad (3.59)$$

But we have

$$\partial_\gamma h_{cd}\partial_b X^\gamma = 2A'(\xi^r)(h_{cd} - \delta_c^r\delta_d^r)\delta_\gamma^r\partial_b X^\gamma \tag{3.60}$$
$$= 2A'(\xi^r)(h_{cd} - \delta_c^r\delta_d^r)\partial_b X^r \tag{3.61}$$
$$= 2A'(\xi^r)(h_{cd} - \delta_c^r\delta_d^r)\delta_b^r \tag{3.62}$$

and

$$\frac{\partial}{\partial(\partial_k X^\gamma)} h_{cd}\partial_b\partial_k X^\gamma = 2g_{\rho\sigma}\partial_c X^\rho \delta_d^k\delta_\gamma^\sigma\partial_b\partial_k X^\gamma \tag{3.63}$$
$$= 2g_{\rho\sigma}\partial_c X^\rho\partial_b\partial_d X^\sigma \tag{3.64}$$
$$= 2\sum_{\rho=1}^{n-1} e^{2A(\xi^r)}\partial_c X^\rho\partial_b\partial_d X^\rho + 2\delta_a^r\partial_b\delta_d^r \tag{3.65}$$
$$= 2\sum_{\rho=1}^{n-1} e^{2A(\xi^r)}\partial_c X^\rho\partial_b\partial_d X^\rho \tag{3.66}$$

hence

$$\partial_b h_{cd} = 2A'(\xi^r)(h_{cd} - \delta_c^r\delta_d^r)\delta_b^r + 2\sum_{\rho=1}^{n-1} e^{2A(\xi^r)}\partial_c X^\rho\partial_b\partial_d X^\rho \tag{3.67}$$

Chapter 4

Conclusion

Through the perspective of gauge-gravity duality, we have considered several properties of quantum fields, roughly grouped by their monotonicty with respect to some parametrization or transformation. We first studied the existence of a Second Law of Thermodynamics in the context of "holographic heat engines," which were constructed from an extension of black hole thermodynamics that includes a prescription for pressure as a function of the cosmological constant of the bulk space. Given an appropriate First Law, we could demonstrate that entropy and temperature were equations of state (*i.e.* functions of pressure and volume); the Clausius Statement and the maximal efficiency of the Carnot engine, or equivalently the Second Law, followed directly from the above, as described in Chapter 2.

The remainder of our work concerned entanglement entropy in the Ryu-Takayanagi description, which conjectures an equality (up to constants) of the entanglement entropy of some spatial subsystem, and the area of the minimal surface incident on the corresponding region of the boundary of the bulk theory in the AdS/CFT picture. We conjectured that given a parameterized chain of spatial subsystems, the entanglement entropy of the subsystems is monotonic with respect to the extension of their respective minimal surfaces into the bulk. Since extension into the bulk is analogous to energy scale, our conjecture constitutes some notion of limited monotonicity over renormalization group flow for entanglement entropy. We term this notion a $\mathcal{S}$-theorem, by analogy to the c-theorem, which conjectures the monotonicity of trace anomaly coefficients with respect to renormalization. Interestingly, the monotonicity also implies that (for sufficiently nice spatial regions), spatial inclusion implies greater entanglement entropy.

Our efforts to demonstrate our conjecture outline a program toward a rigorous result, and include some useful intermediary steps toward that goal. Finally, we consider an ansatz for a family of spherical subsystems in n-dimensional quantum field theories, and attempt to describe equations of motion for locally minimal surfaces of this ansatz. Once completed, a concrete example of minimal surfaces over such a family of subsystems might lend intuition to the more general conjecture, or provide counterexamples to our claims. These results are described in Chapter 3.

4.1 Future work

Our investigation of entanglement entropy and a $\mathcal{S}$-theorem has found many more questions than answers. In particular, there exist a number of conjectures and claims necessary for our argument supporting the existence and monotonicity (with respect to energy scale) of the function $\mathcal{S}$. Several of these are possibly trivial, given a greater knowledge of relevant fields (*e.g.* Claim 3.1.4 would benefit from knowledge of geometric measure theory); others are likely highly specific to the problem at hand. We reproduce these unresolved conjectures below:

Claim 3.1.4. *Let M be a compact, connected, oriented n-dimensional manifold. If f_0, f_1 are embeddings of M in $\mathbb{R}^n$ homotopic under $F : M \times [0,1] \to \mathbb{R}^n$, then denote $f_\phi := F|_\phi$. Define the function $\sigma : [0,1] \to \mathbb{R}$ such that*

$$\sigma : \phi \mapsto \text{area}(\text{Im}(f_\phi)) \tag{3.3}$$

where $\text{area}(U)$ *denotes the Euclidean n-area of the submanifold $U \subseteq \mathbb{R}^n$. Then σ is continuous, and moreover, for any $\phi \in [0,1]$ and $\epsilon > 0$, there exists some neighborhood of ϕ, $N \subseteq [0,1]$, such that for any $a, b \in N$,* $\text{area}(\text{Im}(f_a) \ominus \text{Im}(f_b)) < \epsilon$.

Conjecture 3.1.8. *Let R be a compact $n+1$-dimensional Riemannian manifold with metric $ds^2 = e^{2A(X^r)} dX^\mu dX^\nu \delta_{\mu\nu} + (dX^r)^2$; let E be a constant X^r n-dimensional submanifold E, which is isometric to $\mathbb{R}^n$, and U a compact, connected, oriented n-dimensional submanifold of E. Let $K \subset R$ be any n-dimensional oriented submanifold with $\partial K \subset E$. If $A'(X^r)$ is monotonic, then for any minimal surface V incident on U, if V is strictly enclosed by K, then*

$$\text{area}(V) < \text{area}(K).$$

Claim 3.1.9. *Suppose $V_1, V_2 \subset R$ are n-dimensional manifolds with boundaries $\partial V_1, \partial V_2 \subset E$ such that $V_1 \sim V_2$. Then if $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ such that $W_1 \cup W_2$ is an n-dimensional manifold and $\partial(W_1 \cup W_2) \subset E$, then $W_1 \cup W_2 \sim V_1 \sim V_2$.*

Conjecture 3.1.14. *$\mathcal{R} : [0,1] \to \mathbb{R}$ is continuous.*

Conjecture 3.1.15. *Given $A'(X^r)$ monotonic, $\mathcal{R}$ is injective.*

Resolving the above conjectures is the principle remaining work of the project. Further investigation of the spherical ansatz discussed in Section 3.2 may help facilitate these ends. In particular, considering a lower dimensional case for the ansatz is likely computationally possible, even given only the results above; such an investigation may be fruitful for developing intuition in the more general cases.

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